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Emmanuel Dror Farjoun

Cellular Spaces, Null Spaces and Homotopy Localization

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Introduction

In these notes we describe in some detail a certain framework for doing homotopy theory. This approach emerged in the early 1990's but has roots in earlier work of Bousfield about localization and in the big advances made by Mahowald, Ravenel, Devinatz, Hopkins and Smith towards deeper understanding of the role of periodicity in stable homotopy theory. It is natural to look for a similar unstable organization principle. This has not been found. Rather, certain tools have developed that have proved interesting. In addition, these tools are closely related to the above developments, as well as to central developments that occurred in unstable homotopy with the proof by Miller of the Sullivan conjecture and with the fruitful use of Miller's theorem by Lannes, Dwyer, Zabrodsky and many others.

During these developments the study of homotopy theory through function complexes has become common and productive. Computation of important function complexes has become possible, especially with classifying spaces as domains. It turns out that it is also very productive to formulate localization theory in terms of function complexes. In particular, the notion of a W-null space (essentially, a space X for which the pointed function complex $\mathrm{map}_*(W, X)$ is contractible) has become central in localization theory.

Thus function complexes play a central role in these notes. In fact one can view most of the material as developing techniques that allow better understanding of function complexes not via computing their homology or homotopy groups but directly as spaces. Therefore homotopy colimits become very useful, since it is convenient to have them as domains of function complexes. A typical situation is the decomposition of classifying spaces of compact Lie groups as homotopy colimits by Jackowsky, McClure and Oliver, which allowed a much deeper understanding of function complexes between these objects. In this framework we give an exposition of the work of Bousfield and Thompson about unstable localization and relate it to a better understanding of homological localization.

In relation to homotopy colimits a new tool that comes into play is that of cellular spaces. We show that these structures are closely related to localization, more specifically to colocalizations—homotopy fibres of the localization map. These structures are treated here as being of interest in their own right. They allow one to write, in some interesting instances, classical constructions as pointed homotopy colimits. For example, we examine the symmetric product SP^∞ in this light. This again allows one to better understand function complexes on these spaces which are decomposed as homotopy colimits.

Spaces, function complexes: The present notes can be read either in the category of topological spaces having the homotopy type of CW-complexes, or in the category of simplicial sets. We refer to both as 'spaces' and both categories are denoted by $\mathcal{S}$, or when we talk about pointed spaces as $\mathcal{S}_*$. Although it is perfectly possible to carry out almost the whole theory within the category $\mathcal{S}_*$ of (well-pointed) spaces we do not follow this path, since it is not always the easiest one (see 1.F.7). Rather we mix the discussion of the two categories, pointed and unpointed, trying to avoid the confusion that this might create. The category of simplicial sets is denoted by $\mathcal{SS}$ and that of topological spaces by Top. Often we use the notions of cofibrant and fibrant spaces. In Top cofibrant means (well-pointed) CW-complex while any space in $\mathcal{SS}$ is cofibrant. On the other hand, every topological space is fibrant while fibrant in $\mathcal{SS}$ means a simplicial set that satisfies the Kan extension condition [Q-1], [May-1]. Whenever some construction in Top, especially those involving mapping spaces, yields a non-CW space we can and do pull them back to the class of CW-spaces via the canonical CW-approximation (compare e.g. (1.B) or (1.F)). By a finite space we mean finite CW-complex or a simplicial set with a finite number of non-degenerate simplices.

Since we make extensive use of function complexes, care must be taken that simplicial sets that serve as ranges in function complexes are fibrant, satisfying the Kan extension condition [May-1], while spaces that serve as domains are always assumed to be cofibrant. Otherwise the homotopy type of a function complex is not invariant under weak equivalence and has in general no homotopy meaning. When we write $\text{map}_*(X, Y)$ or $\text{map}(X, Y)$ in the topological category we most often use only the underlying weak homotopy type of the space of continuous maps (pointed or unpointed), so there is no need to turn it into an internal function complex having the homotopy type of a CW-complex. For typographical reasons the notation Y^X is often used to denote the function complex of maps from X to Y. We denote by $\simeq$ a weak homotopy equivalence. Certain constructions though are easier to handle in the category of simplicial sets where $\text{map}(X, Y)$ denotes the usual simplicial function complex [M-1]. It is often possible to carry over the necessary construction naturally into topological spaces using the pair of adjoint functors, the realization and singular functors. This is demonstrated in some detail in section 1.F.

A note about chapters and sections: References within the nine chapters are by sections, such as (B.3.5). When referring to other results or sections outside the current chapter, the number of the chapter precedes that of the section or result, e.g. (1.F.6.1) is a result or a figure from Chapter 1, section F.

Some details about the contents: In Chapter 1 the basic notions of f-local space and f-localization with respect to an arbitrary map denoted $\mathbf{L}_f$, are introduced. A special case, when the map f is null homotopic, has particularly pleasant properties and is called nullification, denoted by $\mathbf{P}_A$, when the map is $A \to *$. This last functor allows one to introduce an interesting partial order on spaces that is analyzed later on: one says that X 'supports Y' or 'kills Y', denoted by: $X < Y$, if $\mathbf{P}_X Y \simeq *$. This is really the same as the implication: For any space T, $\mathrm{map}_*(X,T) \simeq * \Rightarrow \mathrm{map}_*(Y,T) \simeq *$ (note, however, the different convention-notation followed in [B-4] where the sense of $<$ is reversed.)

We give a list of elementary properties of localization that forms the beginning of a sort of localization calculus, which will allow one to control the behavior of $\mathbf{L}_f$ under standard homotopy operations such as suspensions, loops, and homotopy colimits. These functors are universal in two senses: they are both terminal and initial up to homotopy in certain classes of maps. Still we do not know of any inverse limit constructions that present them as initial objects analogous, say, to the Bousfield–Kan construction of their localizations as an inverse limit.

We also begin to note some crucial properties that distinguish the nullification from the general localization. In particular the following seems to be a basic distinction:

When $\mathbf{P}_A$ is applied to the homotopy fibre of the coaugmentation map $X \to \mathbf{P}_A X$ one always gets a point up to homotopy: that is, there is a universal equivalence $\mathbf{P}_A(\mathrm{Fib}(X \to \mathbf{P}_A X)) \simeq *$. The analogous formula for $\mathbf{L}_f$ is weaker.

Chapter 2 can be seen as an attempt to discuss more carefully the homotopy fibre of the nullification map. We now know that this homotopy fibre when considered as a functor on the pointed category of spaces is an idempotent augmented functor denoted by $\overline{\mathbf{P}}_A$. It is a sort of colocalization. Since $\mathbf{P}_A X$ is really X stripped of all its 'A-information' the homotopy fibre $\overline{\mathbf{P}}_A X$ still contains all this information, and in fact $\mathrm{map}_*(A, \overline{\mathbf{P}}_A X)$ is equivalent to $\mathrm{map}_*(A, X)$. But in general $\overline{\mathbf{P}}_A X$ is not the universal space with this property.

There is another canonical space denoted by $\mathbf{CW}_A X$, which is the universal space having the same function complex from A as X. Furthermore, this space is built out of copies of A and approximates X much in the same way that a classical CW-approximation (which is 'composed of cones on spheres' and extracts the 'spherical information' from X expressed in the usual form of the homotopy groups) gives a 'spherical approximation' to X. Thus we consider here a second partial order, denoted by $\ll$, which, as it turns out, is closely related to $<$ defined above: namely, $X \ll Y$ if and only if the pointed space Y can be built from the pointed space X by repeatedly applying, say, wedges and homotopy pushouts, possibly infinitely many times. We say that Y is *X-cellular* in that case and we begin to consider the above cellularization functor and this partial order in this chapter. For example, one shows that a finite product of X-cellular spaces is X-cellular (2.D.16). Notice that if $X \ll Y$, and X is acyclic with respect to any homology theory, then so is Y (2.D.2.4). Also we shall see that $X \ll Y$ always implies $X < Y$.

In particular we begin to develop criteria to decide when a given space X is A-cellular with respect to another space A, i.e. under what conditions the equivalence $X \simeq \mathbf{CW}_A X$, or equivalently $A \ll X$, holds. Such criteria are an important concern in these notes. This is handy when, for example, one wants to know under which conditions a K-acyclic space can be constructed by pushouts and telescopes from elementary K-acyclic spaces such as the cofibre of the Adams map, and is therefore the direct limit of its K-acyclic finite subspaces.

We then see that we have obtained two seemingly closely related functors. One would like to show, but it is not yet known how to proceed in all cases, that these two idempotent functors, the localization $\mathbf{L}_f$ and cellularization $\mathbf{CW}_A$, are in fact two facets of a symmetric construction that factors an arbitrary map $X \to Y$ into a 'cofibration' followed by a 'trivial fibration': One can change the usual notions of weak equivalences in the 'standard model category' of spaces by, say, adding a single map f to the class of weak equivalences, but this will change also the notion of fibre maps along which one should be able to lift weak equivalences that are cofibrations. The localization would then be just factorization of the map $X \to *$ while cellularizations are factorizations of the map $* \to Y$. These observations put the above functors in a reasonable theoretical light. Hirschhorn is developing these directions carefully in a general framework [HH]. We then continue to show that certain standard constructions lead to cellular relations: For example, the cellularity of the third term in a fibration sequence can be predicted when one knows the other two.

In Chapter 3 we turn to deeper technical properties of these idempotent functors: the rule of commutation with the loop space functor and in general with taking the homotopy fibre of a map. It turns out that one has general formulas $\mathbf{L}_f\Omega \simeq \Omega\mathbf{L}_{\Sigma f}$ and $\mathbf{CW}_A\Omega \simeq \Omega\mathbf{CW}_{\Sigma A}$. These are a fundamental part of the calculus of localization and are used to show, for example, that fibrations over a W-null base space are preserved by nullification with respect to the suspension of W, but also much more general theorems concerning preservation of fibrations.

These formulas also allow us to compute directly, and formally, certain cellular relations, such as the fact that $\Omega\Sigma X$, the James construction on X, is always X-cellular. In fact one can show that $\Sigma X < Y \Rightarrow X \ll Y$.

Chapter 4 serves two purposes. First, we present a more careful analysis of pointed homotopy colimits and their relations to the usual strict colimits (direct limits) of diagrams of spaces. This allows us to show, for example, that $SP^\infty X$, the Dold–Thom symmetric product on X, which initially, like the James construction, is defined as a strict colimit via a point-wise construction, can in fact be built by a pointed homotopy colimit starting from the initial space X alone. Since by the Dold–Thom theorem the infinite symmetric product is a GEM, i.e. a product of

Eilenberg–Mac Lane spaces, and is, in fact, the universal GEM associated with X, its expression as a homotopy colimit allows one to understand better the operation of localization and cellularization on generalized Eilenberg–Mac Lane spaces. This paves the way to the second purpose: a cellular version of a 'key lemma' of Bousfield. This version, following an approach taken by Dwyer [Dw-2], describes the cofibre of the map from the Borel construction on X to the corresponding strict quotient space. The key lemma is related to a cellular estimate of the cofibre as being $\Sigma^2 X$-cellular. This means, roughly speaking, that in order to build $\Sigma SP^\infty X$ from ΣX, exactly one copy of ΣX is needed, and then only higher suspensions of X.

In Chapter 5 we show that if in a fibration sequence $F \longrightarrow E \longrightarrow B$, $\mathbf{P}_{\Sigma A}$ kills both the base space and the total space, i.e. $\mathbf{P}_{\Sigma A} B \simeq *$ and $\mathbf{P}_{\Sigma A} E \simeq *$, it may not kill the fibre F, but it always turns it into an 'homotopy abelian object': we show that $\mathbf{P}_{\Sigma A} F$ is naturally an infinite loop space that is equivalent as such to a product of Eilenberg–Mac Lane spaces with their usual abelian infinite loop space structure. This is done using the results of Chapter 4, and notably Bousfield's key lemma and the 'infinite loop space machine' of Segal, as well as substantial parts of [DF-S], extending their results to cofibrations and the cellular approximation functor.

This approach leads to a general theorem about the preservation of fibration by the nullification 'up to an abelian error term'. We then use the relation between localization with respect to a map and nullification with respect to its cofibre to deduce a general theorem about the localization of arbitrary fibration with respect to any double suspension. It is perhaps worth mentioning here that the fact that the classical Sullivan type of localization preserves fibrations over, say, a 1-connected space is a special case of these preservation-of-fibrations theorems: from the present point of view, the Sullivan localization of a simply-connected space with respect to a prime p is just the Anderson localization [An], which is in turn simply nullification with respect to $\Sigma M^2(p)$, the suspension of the two-dimensional Moore space. Similar reasoning is then applied in examining the effect of applying $\mathbf{CW}_A$ to a fibre sequence, with results that are weaker, but similar to the above.

We then apply this theory to show two remarkable examples: first, we describe a theorem of Neisendorfer which states that any finite 2-connected complex can be recovered, up to p-completion, from its n-connected cover, for any $n \geq 0$. This is much stronger than saying that these spaces must have non-trivial homotopy in infinite dimensions: it shows that somehow this 'infinite tail' has all the information needed to reconstruct the 'lower dimensional information'. Secondly, we show that this Serre-type result about homotopy groups in high dimensions generalizes to infinite spaces, too: their A-homotopy groups must be non-trivial in infinitely many dimensions, as long as these spaces are built by a finite number of A-cells from $\Sigma^i A$, where $i \geq 1$.

In Chapter 6 we turn our attention to homological localization with respect to generalized homology theories, such as Morava K-theory. We essentially reproduce the relevant material from [DF-S], showing that short of a small error term these localizations preserve twice looped fibrations. It is reasonable to expect that this is also the case for single-loop-space fibrations.

Note, however, that in order to present homological localization in the form $\mathbf{L}_f$, for some map f, one needs to take a 'monster map': in general, one must take the union of all E_*-homology isomorphisms between spaces whose cardinality is not bigger than the coefficients of E_*.

While theoretically this can be done, it is certainly desirable to replace this 'monster map' with a smaller object. To do that so, one considers a classification of possible nullification functors, under some restrictions in Chapter 7. This means roughly the classification of all possible nullification functors with respect to finite p-torsion suspension spaces. The above mentioned nullity classes of spaces define a very rough equivalence on spaces, namely, W and V are null-equivalent or of the same nullity class if, for any pointed space T, one has the double implication: $\text{map}_*(V,T) \simeq *$ if and only if $\text{map}_*(W,T) \simeq *$. This really means that the functors $\mathbf{P}_W$ and $\mathbf{P}_V$ are naturally equivalent. The classification of these classes, starting with similar but much easier stable classes, is undertaken in Chapter 7. It turns out, making heavy use of Bousfield's theory of fibrations above, that the stable and unstable classifications are not very different from each other, and the main invariant needed here is the stable one, namely, the 'Hopkins–Smith type' related to Morava K-theories. This possibly is not all that surprising since, if one localizes with respect to a nilpotent self-map $\Sigma W \to W$, one obtains the same results as nullification with respect to W and, by [D-H-S], there is essentially only one self-map on the above complexes that is not nilpotent. Using the classification of nullity classes one can also classify the closely related cellular classes of the above suspension spaces.

The classification of nullity classes and possible nullification functors can be used to analyse higher periodicity. This is done in Chapter 8 with respect to v_1. We follow the work of Bousfield and Thompson in regard to K-theory localization. But we use the cellular analysis to express the nullification and K-localization functors in more elementary terms, as a telescope of the Adams map. Thus in this case, modulo some technicalities, the above monster map used to express K-localization can be replaced by a single map between two Moore spaces. This is a sort of ideal situation that could hold, in general, if a kind of 'unstable telescope conjecture' were true.

The basic result is that one can express the function complex $\text{map}_*(M, \mathbf{P}_{V(1)}E)$ as a mapping telescope of function complexes, which inverts v_1

in the most elementary way: namely, as

$$\mathrm{Tel}\Big(E^M \to E^{\Sigma^q M} \to E^{\Sigma^{2q} M} \to \cdots\Big)$$

where M denotes the appropriate Moore space. This latter telescope is of course neither idempotent nor coaugmented as a functor, so it cannot replace the localization in general, but it still captures in a direct way the v_1-periodic homotopy. In particular its homotopy groups depend only on the action of the v_1 operator on $\pi_* E$ as a graded group.

As a corollary one can explain to what extent higher loop spaces on K-acyclic spaces are still acyclic. In addition one can show that K-acyclic, p-torsion spaces whose loop spaces are also K-acyclic can be built via cofibration sequences from the cofibre of the Adams map v_1.

In the final chapter, Chapter 9, we develop several tools that allow us to detect and prove interesting cellular inequalities. The basic idea here is the passage from *pointed* homotopy colimits over arbitrary indexing categories to *unpointed* homotopy colimits over categories with a contractible classifying space. Explaining ideas from [Ch-2] and [DF-5], the program here is to show that often the well-known theorems on connectivity of homotopy constructions such as homotopy fibre can be strengthened to a theorem asserting the cellularity of these constructions. On occasion this gives new connectivity results too. A typical result in this direction is that the fibre of the cofibration quotient map $X \to X/A$ is always A-cellular. Furthermore the homotopy fibre of a map to connected space can be built by pointed homotopy colimits from the collection of the actual inverses of points (say barycentres) in the base space.

We conclude with some applications of this technique: In particular one can show that while there is no easy relationship between the homotopy limit and colimits of diagrams, some inequalities can be proven in general between these constructions. These form a sort of generalization of the elementary inequality $\Sigma\Omega X \ll \Omega\Sigma X$.

Acknowledgments: Several crucial ideas that I have tried to explain here were developed by A.K. Bousfield in his paper [B-4] about unstable periodicity. In particular he proved there the key lemma that opens the way to many of the more interesting results about localizations. This work was initiated while I was visiting the University of Geneva and the E.T.H. research institute and I am grateful for their hospitality. A good part of the present work was done while I was visiting Purdue University. I am very grateful for their hospitality and the close cooperation in particular with Jeff Smith with whom [DF-S], which is also a part of the present notes, was written. I am very grateful to W. Dwyer for his help in deciphering the relations of the key lemma to homotopy coends and his proof of (3.C.16). His support in the project of understanding cellular spaces was very useful. It was partly done in the framework of a research grant from the Binational Science Foundation. No less useful was the close cooperation on the initial chapters with Hirschhorn who is now developing the theoretical foundations and model theoretical development in a separate monograph. Dwyer's student W. Chacholsky has taken an interest in these problems and greatly helped in developing ideas about cellularity of homotopy fibres in Chapter 9. The referees' comments were very valuable and helpful. I am also very grateful to TEX specialist Simcha Kojman who was very patient with my changes; she and copy editor Paul Greenberg greatly facilitated the final production.

Contents

1. COAUGMENTED HOMOTOPY IDEMPOTENT LOCALIZATION FUNCTORS

Introduction

In this chapter basic notions that will be used throughout the present notes are defined, in particular that of f-local spaces and f-equivalences between spaces. A list of elementary properties of the basic notions and of localization is given in section 1.A.8 below. These properties mostly follow easily and directly from the definitions and are used in many arguments. A construction of the f-localization functor is given and its property of continuity is discussed. This is useful but not essential in constructing a fibrewise version of localization. Continuity also renders certain induced maps such as $\mathrm{aut}(X) \longrightarrow \mathrm{aut}(\mathbf{L}_f X)$ easily understandable, where aut denotes the spaces of self-homotopy equivalences.

We pay some attention to the localization of homotopy colimits. The interested reader can find a brief discussion of these colimits in Appendix HL below. The fact that localization behaves relatively well under homotopy colimits–including wedge sum, for example–is very helpful later on. We then show that well-known localization functors including e.g. the Quillen plus construction are special cases of this general homotopy localization. Then there is a discussion of fibrewise localization and several approaches are discussed. This discussion is carried out in a bit more general framework of applying homotopy functors fibrewise. We show how to do that using homotopy colimits under mild assumptions on the functor. As a first application of these fibrewise localizations one deduces two very useful properties of the homotopy fibre of the localization map: first we show that if the localization kills the fibre then it preserves the fibration. Then we show that the localization with respect to a null map always kills the homotopy fibre of the localization map.

A. Local spaces, null spaces, localization functors, elementary facts

We consider here the notion of f-local space where $f : A \to B$ is an arbitrary cofibration map between cofibrant spaces (i.e. CW-complexes if we work in Top). Bousfield in [B-2] has already shown how to associate an f-local space $\mathbf{L}_f X$ with any space X together with a coaugmentation map $X \to \mathbf{L}_f X$. It turns out that in spite of its generality, this localization functor has many useful properties that combine to form a 'calculus of localization'. We examine in this chapter some of these most basic properties. In case the cofibration f is a null homotopic map, the functor $\mathbf{L}_f$ has stronger and cleaner properties and is called *nullification* (with respect to $A \vee B$, see below).

In practice, all the known coaugmented homotopy functors **F** which are also idempotent (i.e. roughly **FF** is equivalent to **F**) have the form $\mathbf{L}_f$ for a suitable f, so the present framework and result may well apply to any idempotent functor. Notice, however, that Bousfield–Kan's R_∞ is not in general idempotent.

In this chapter we also consider somewhat more delicate properties of localization, in particular its value on homotopy colimits and on the homotopy fibre of the localization (coaugmentation) map.

A.1 DEFINITION : *(f-local, W-null): We say that Y is f-local (where f is a map $f : A \to B$ between cofibrant spaces) if Y is fibrant and the map f induces a weak homotopy equivalence on function complexes,*

$$\mathrm{map}(f, Y) : \mathrm{map}(B, Y) \xrightarrow{\simeq} \mathrm{map}(A, Y).$$

In case the map is simply $w : * \to W$ one refers to a w-local space Y as W-null; this means that the natural map $Y \xrightarrow{\simeq} \mathrm{map}(W, Y)$ is an equivalence. Equivalently one defines these concepts in the pointed category of spaces (where now all spaces are assumed to be well-pointed): A fibrant space is local if the corresponding map of function complexes of pointed maps is a weak equivalence

$$\mathrm{map}_*(f, Y) : \mathrm{map}_*(B, Y) \xrightarrow{\simeq} \mathrm{map}_*(A, Y).$$

Remark: The fibration $\mathrm{map}_*(V, X) \longrightarrow \mathrm{map}(V, X) \longrightarrow X$ for any cofibrant V over any connected and fibrant X shows that for a connected and fibrant space X the map induced by f, namely $\mathrm{map}(f, Y)$, is an equivalence iff the map $\mathrm{map}_*(f, Y)$ is an equivalence with respect to any choice of ('well-pointed') base points.

A.1.1 EXAMPLES: We give examples in sections E and 2.D below. Here we note several quick illustrations: If the map f is the map of the n-sphere to a point $* \longrightarrow S^n$, then an f-local connected space is an S^n-null connected space, i.e. it is a space X whose n-th loop space $\Omega^n X$ is contractible. Thus such a space has no homotopy groups above dimension $n-1$ and is otherwise arbitrary. Thus it is just an arbitrary Postnikov $(n-1)$-stage. Dually it is easy to see that a space X is n-connected if and only if any Eilenberg–Mac Lane space $K(G, i)$ for $0 \leq i \leq n$ is X-null. For a more difficult example, if the map g is the degree p map from the n-sphere to itself, then a *connected, pointed* space X is g-local if the map on its n-th loop space raising every loop to its p-th power is a weak homotopy equivalence of the underlying spaces, disregarding the loop structure. For $n > 1$ this means that all the homotopy groups above dimension $(n-1)$ are uniquely p divisible, see E.3 below.

A.1.2 REMARK: One might ask why we do not define a 'homotopy f-local' space to be a space W for which the induced map on homotopy classes of maps (rather than on the full function complexes), namely the map $[f,W] : [B,W] \to [A,W]$ is an isomorphism of sets. This is a perfectly good definition, but it turns out that it too leads directly to the definition given above. The reason is that given a notion of 'f-local' space we are mostly interested in functors that turn an arbitrary space into an 'f-local' one. Now the following fact shows that as far as functorial constructions are concerned the definition using homotopy classes leads to one in which the full function complexes are used.

A.1.3 FACT: For any continuous (or simplicial, see C.8 below), idempotent, coaugmented functor F : {Spaces} $\to$ {Spaces}, if, for all X, the induced map on homotopy classes $[f, FX]$ is an isomorphism of sets, then FX is automatically f-local: $\text{map}(f, FX) : \text{map}(B, FX) \xrightarrow{\sim} \text{map}(A, FX)$ is an equivalence.

This is Corollary (1.3) in [DF-4] which was written in view of this and similar questions.

Another way of viewing Fact A.1.3 is to notice that it implies that it is impossible to canonically associate a universal 'homotopy f-local' space with every space X.

This is best understood by an example (due to G. Mislin): Let $f : S^1 \to *$, then a 'homotopy f-local' space is just a simply connected space. We ask: Is there an initial object among all maps of a space, say of $\mathbb{R}P^2$, to 1-connected spaces? The answer is NO. To see why, notice that by unique factorization up to homotopy such a space U would need to have $H^2(U, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ since the non-trivial map $\mathbb{R}P^2 \to \mathbb{C}P^\infty$ would also have two factors through U, uniquely up to homotopy. But U is 1-connected and its second cohomology cannot have torsion.

A.2 DEFINITION: *A functor* $\mathbf{F}$ *is called* coaugmented *if it comes with a natural transformation* Id $\to \mathbf{F}$, *i.e. for each* $X \in \mathcal{S}$ *a natural map* $j_X = j : X \to \mathbf{F}X$. *A coaugmented functor* $\mathbf{F}$ *is said to be idempotent if both natural maps:* $\mathbf{F}X \rightrightarrows \mathbf{FF}X$, *namely both* $j_{\mathbf{F}X}$ *and* $\mathbf{F}(j_X)$, *are weak equivalences and are homotopic to each other. We say that the coaugmentation map* j_X *is* homotopy universal *with respect to maps* $X \to T$ *into* f*-local spaces* T *if any such map factors up to homotopy through* $X \to \mathbf{F}X$ *and the factorization is unique up to homotopy.*

The next few pages will present a construction of localization functor [B-2] [DF-2] [C-P-P]:

A.3 THEOREM: *For any map $f : A \to B$ in $\mathcal{S}$ (or $\mathcal{S}_*$, see remark A.7) there exists a functor $\mathbf{L}_f$, called the f-localization functor, which is coaugmented and homotopically idempotent. Any two such functors are naturally weakly equivalent to each other. The map $X \to \mathbf{L}_f X$ is a homotopically universal map to f-local spaces. Moreover, $\mathbf{L}_f$ can be chosen to be continuous or simplicial in the sense explained (1.C) below.*

Proof: The construction of $\mathbf{L}_f$ is carried out in section B below. The proofs of claims about $\mathbf{L}_f$ are in (B.5), (C.1), (C.2), and (C.12) below.

A.4 NULLIFICATION FUNCTORS $\mathbf{P}_W$, NULLITY CLASSES: A special role is played by localization with respect to maps of the form $W \to *$, or $* \to W$. In that case a pointed and connected space X is $(W \to *)$-local or, by (A.1), W-null if and only if $\text{map}_*(W, X) \simeq *$ or $\text{map}(W, X) \simeq X$. The localization with respect to these null maps deserves a special name due to its much better behavior and common occurrence. One denotes the localization $\mathbf{L}_{W \to *} = \mathbf{L}_{* \to W}$ by $\mathbf{P}_W$; we call $\mathbf{P}_W$ the W-nullification functor. Bousfield used the term W-periodization for $\mathbf{P}_W$. It plays a major role in his theory of unstable periodic homotopy, as we shall see below. This notation also emphasizes the affinity of general nullification functors to their early predecessor, the Postnikov section functor P_n that we saw above.

Of course, the condition $\text{map}_*(W, X) \simeq *$ occurs often in homotopy theory especially since the proof by Miller of the Sullivan conjecture, that says in these terms that any finite-dimensional space is $K(\pi, 1)$-null for any locally finite group π. The concept of trivial function complex plays a major role in the present notes and we use it right away to define a useful partial order on pointed or unpointed spaces.

A.5 NULLITY CLASSES, (WEAK) PARTIAL ORDER $X < Y$: We say that X supports Y or that Y is X-supported and denote it by $X < Y$ if any X-null space is also Y-null. This is a transitive but not anti-reflexive relation. It is equivalent as we shall see to $\mathbf{P}_X Y \simeq *$ (A.8)(e.9) below. One says that X and Y have the same nullity (class) if $X < Y$ and also $Y < X$. Thus $S^n < S^{n+1}$ and $X \vee X$ has the same nullity as X. Notice that [B-4] uses the opposite convention in the notation of the partial order

A.6 EXAMPLE: $P_{S^{n+1}}$ is the n-th Postnikov section $P_n X$ which can be characterized by $\Omega^{n+1} P_n X \simeq *$. Compare A.1.1 above. An important result of Zabrodsky and Miller [M], a strong version of which is given in (2.D.13) below, says in this notation that for any topological group G one has $G < BG$. In fact we shall prove the sharper inequality: $\Sigma G < BG$ is always true. See (9.D.4) below.