BASIC LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Preface

The discrepancy between what is taught in a standard course on partial differential equations and what is needed to understand recent developments in the theory is now very wide. It is a fact that only a relatively small number of specialists, in a few universities, are able, these days, to teach a course that is truly introductory to those developments. Perhaps this is not much different from what has been happening in all active areas of mathematics. But it is also true, speaking of the best graduate students, as well as of professional mathematicians, that when they are said to be conversant in all aspects of mathematics, this often excludes substantial portions of analysis and most of partial differential equations.

The complementary facet of such a state of affairs is that many up-to-date expositions fail, frequently because of lack of time, to show the link with the older results, and give the erroneous impression that the modern theories have no roots and are cut off from a rich past. The truth, of course, is that progress comes not only from pushing further and further into new territory but also from frequent returns to the familiar grounds, from seeking an ever-deeper understanding of their nature, and finding there new inspiration and guidance.

The archetypes of linear partial differential equations (Laplace's, the wave and the heat equations) and the traditional problems (Dirichlet's and Cauchy's) are the main topic of this book. Most of the basic classical results can be found here. But the methods by which these are arrived at are definitely not traditional; the methods are, in practically every instance, applications of those now in favor at a higher level of abstraction. The aim of this approach is twofold: it is, on one hand, that of recalling the classical material to the modern analyst, in a language he can understand; on the other hand, that of exploiting the same material, with the wealth of examples it provides, as an introduction to the modern theories.

Developments toward greater generality have not been avoided when it was felt that they represented the natural "next step" and afforded a meaningful opening to the more advanced stages of the theory—provided, also, that they did not require more machinery than had been made available up to that point. Thus the reader will find a discussion of the Cauchy problem for first-order systems of hyperbolic equations with constant coefficients in Section 15, following the study of the same problem for the wave equation. Similarly, Gårding's inequality for strongly elliptic equations of any (even) order is established in Section 36, and, in a somewhat more philosophical vein, the meaning of the Lopatinski boundary conditions is explained in Section 38.

The approach to the classical Dirichlet problem calls for some comment. Because I felt committed to describe the classical results, it was out of the question to limit the discussion to the weak solution, or variational, method—even strengthened by the proof of regularity up to the boundary, when the latter is sufficiently smooth. After all, one might want to have the solution to the Dirichlet problem in a cube when the boundary value is continuous. Thus I was resigned to the dichotomy between the variational methods within the framework of the Sobolev spaces, and the Perron–Brelot method, tied to potential theory, until Guido Stampacchia indicated to me how to make the transition from the former to the latter, by way of his weak maximum principle (Section 28). I have followed his advice and adapted the argument of his article [2] (where, needless to say, more general second-order elliptic equations than Laplace's are studied). From there on, the classical potential theory can easily take off, as is succinctly indicated in Sections 29 and 30.

Like potential theory, many other important topics are very lightly touched upon: for example, the Dirac equations, random walks, the finite difference method, and continuous semigroups of operators. Here the book is truly introductory; its sole ambition is to give an idea of what these topics are all about and a taste for learning more. Thus it is not a treatise. Nor is it a classroom text, due to its size and the quantity of its contents, although it is true that it began as a set of lecture notes used at the University of Miami and at Rutgers University. Some readers might find that the writing shows too little regard for concision—for which I apologize. I have made a point, rather, of explicitly formulating some of the many thoughts that usually go unformulated while writing, especially while writing mathematics; and I hope not to have totally failed in this.

Today, distributions are the language of linear PDE theory, and I am certainly not of the school that would like to do without them. But knowing that not all students are seriously exposed to distributions, I have limited their use to their more mechanical aspects—convergence of sequences, differentiation, convolution; sometimes, but not often, the local representation of a distribution as a finite sum of derivatives of continuous functions is used to advantage. Fourier transformation of distributions, however, is used systematically; the student genuinely interested in PDE must make an effort to learn it. Not that much effort is needed, for it is such a smooth and simple theory: Excellent expositions are found in Chapter VII of L. Schwartz' book [TD], or in Chapter I of Hörmander's book [LPDO] (and in many other texts). In particular, the reader should be familiar with the Plancherel theorem and with the Paley–Wiener(-Schwartz) theorem. As far as linear functional analysis is concerned, the basic facts about Hilbert and Banach spaces must be known, but nothing much deeper—although, from the middle of the book on, an ever greater use is made of functions (and, later, distributions) with values in Banach spaces. Finally, it is presumed that the student has a fairly good knowledge of holomorphic functions of one complex variable, of real variable theory, mainly Lebesgue integration, and a smattering of measure theory. A bit of linear algebra will be of help, here and there.

There are 390 exercises, and several contain detailed information which should enable the reader to reconstruct the proofs of some important results: for example, the hypoellipticity of elliptic equations—of any order—with C^∞ coefficients, in Exercises 36.4 and 36.7, or the theorem of supports—in one variable—in Exercises 43.4, 43.5, and 43.6. Other exercises are simple variants or straightforward applications of the results and the methods in the text.

Notation

$\mathbf{R}^n$ product of n copies of the real line $\mathbf{R}$
$x^1, \ldots, x^n$ coordinates in $\mathbf{R}^n$; also $y^1, \ldots, y^n$, etc.
$x = (x^1, \ldots, x^n)$ the variable in $\mathbf{R}^n$; also y, etc.
$\mathbf{R}_n$ *dual* of $\mathbf{R}^n$
$\xi_1, \ldots, \xi_n$ coordinates in $\mathbf{R}_n$; also $\eta_1, \ldots, \eta_n$, etc.
$\xi = (\xi_1, \ldots, \xi_n)$ the variable in $\mathbf{R}_n$; also η, etc.
$x \cdot \xi = x^1\xi_1 + \cdots + x^n\xi_n$ the scalar product between a vector in $\mathbf{R}^n$ and a covector in $\mathbf{R}_n$; also $\langle x, \xi \rangle$
$\mathbf{Z}^n$ product of n copies of the set $\mathbf{Z}$ of integers (of all signs)
$\mathbf{Z}^n_+$ product of n copies of the set $\mathbf{Z}_+$ of nonnegative integers
$\mathbf{C}^n$ product of n copies of the complex plane $\mathbf{C}$
$z^1, \ldots, z^n$ complex coordinates in $\mathbf{C}^n$
$z = (z^1, \ldots, z^n)$ the variable in $\mathbf{C}^n$
$\operatorname{Re} z = (\operatorname{Re} z^1, \ldots, \operatorname{Re} z^n)$ the *real part* of the complex vector z
$\operatorname{Im} z = (\operatorname{Im} z^1, \ldots, \operatorname{Im} z^n)$ the *imaginary* part of z
$\mathbf{C}_n$ dual of $\mathbf{C}^n$
$\zeta_1, \ldots, \zeta_n$ coordinates in $\mathbf{C}_n$
$\zeta = (\zeta_1, \ldots, \zeta_n)$ the variable in $\mathbf{C}_n$
$z \cdot \zeta = z^1\zeta_1 + \cdots + z^n\zeta_n$ the (real) scalar product between $z \in \mathbf{C}^n$ and $\zeta \in \mathbf{C}_n$; also $\langle z, \zeta \rangle$
$|x| = \{(x^1)^2 + \cdots + (x^n)^2\}^{1/2}$ the *Euclidean norm* on $\mathbf{R}^n$
$|\xi|, |z|, |\zeta|$ the Euclidean norms on $\mathbf{R}_n$, $\mathbf{C}^n$, $\mathbf{C}_n$, respectively
$|\alpha| = \alpha_1 + \cdots + \alpha_n$ the *length* of the n-tuple $\alpha \in \mathbf{Z}^n_+$
$\alpha! = \alpha_1! \cdots \alpha_n!$, $\binom{\alpha}{p} = \binom{\alpha_1}{p_1} \cdots \binom{\alpha_n}{p_n}$, $\binom{\alpha_j}{p_j} = \alpha_j!/\beta_j!(\alpha_j - \beta_j)!$ where $\alpha, \beta \in \mathbf{Z}^n_+$ and $\alpha_j \geq \beta_j$ for every $j = 1, \ldots, n$
$d(x, A)$ Euclidean distance from the point x to the set A
$B_r(x)$ open ball centered at x, having radius r
sup *supremum*, or least upper bound, of a set of real numbers
inf *infimum*, or greatest lower bound, of a set of real numbers
ch A *convex hull* of a set A (contained in a linear space)
Ω an open subset of $\mathbf{R}^n$
$[a, b]$ a *closed* interval, with limit points a and b in the real line $\mathbf{R}^1$ (also when either $a = -\infty$ and/or $b = +\infty$)

$[a, b[$ semiclosed interval $a \leq t < b$; $]a, b] = \{t \in \mathbf{R}^1; a < t \leq b\}$
$]a, b[$ open interval $a < t < b$
$\complement A$ complement of A
$A \backslash B$ complement in A of the subset B of A
$a \mapsto f(a)$ mapping which to the object a assigns the value $f(a)$
$f: E \to F$ mapping f, defined in the set E, and valued in the set F
supp u the *support* of u (smallest closed set outside which $u \equiv 0$)
$\dfrac{\partial u}{\partial x^j}$ partial derivative of u with respect to x^j; also u_{x^j}, $\partial_{x^j} u$; also u' if there is only one variable

$D_j = -\sqrt{-1}\,\dfrac{\partial}{\partial x^j}$; also D_{x^j}

$(\partial/\partial x)^\alpha = (\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^n)^{\alpha_n}$, $(\alpha \in \mathbf{Z}^n_+)$
$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$
$dx = dx^1 \cdots dx^n$ the *Lebesgue measure* on $\mathbf{R}^n$; $d\xi = d\xi_1 \cdots d\xi_n$ the analog in $\mathbf{R}_n$ (both measures dx and $d\xi$ assign the volume 1 to the unit cube)
dz^1 the *line measure* in the complex plane (oriented counterclockwise)
$dz = dz^1 \cdots dz^n$ the product of n line measures in $\mathbf{C}^n$

$(f * g)(x) = \int_{\mathbf{R}^n} f(x - y)g(y)\, dy$ the value at x of the *convolution* of f and g

$\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} u(x)\, dx$ *Fourier transform* of u $(i = \sqrt{-1})$

$\mathscr{F}u$ also Fourier transform of u

$\mathscr{F}^{-1}v(x) = (2\pi)^{-n} \int_{\mathbf{R}_n} e^{ix\cdot\xi} v(\xi)\, d\xi$ *inverse Fourier transform* of v

$P(x, \partial/\partial x) = \sum_{|\alpha| \leq m} c_\alpha(x)(\partial/\partial x)^\alpha$ a linear partial differential operator of order m in Ω (generally with C^∞ coefficients c_α)

${}^tP(x, \partial/\partial x) = \sum_{|\alpha| \leq m} (-\partial/\partial x)^\alpha c_\alpha(x)$ the (*formal*) *transpose* of $P(x, \partial/\partial x)$

$P(x, \partial x)^* = \sum_{|\alpha| \leq m} (-\partial/\partial x)^\alpha \overline{c_\alpha(x)}$ the (*formal*) *adjoint* of $P(x, \partial/\partial x)$

$\Delta = \left(\dfrac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\dfrac{\partial}{\partial x^n}\right)^2$ the Laplace operator (or Laplacian) in n variables

$\square = \left(\frac{\partial}{\partial t}\right)^2 - \Delta$ the wave operator (or *d'Alembertian*) in *n space* variables (t always denotes the *time* variable)

E, **F**, ... Hilbert spaces or Banach spaces over the complex numbers (sometimes over the real numbers)

x, **e**, ... elements of the Banach space **E**

$\|\mathbf{e}\|_{\mathbf{E}}$ *norm*, in the Banach space **E**, of the element **e**

$L(\mathbf{E}; \mathbf{F})$ space of bounded linear operators of **E** into **F**, equipped with the *operator norm*

$$\|A\| = \sup_{0 \neq \mathbf{e} \in \mathbf{E}} \|A\mathbf{e}\|_{\mathbf{F}} / \|\mathbf{e}\|_{\mathbf{E}}$$

$\mathbf{E}'$, $\mathbf{E}^*$ *topological dual* of the Banach space **E**, the space of continuous linear functionals on **E** [equipped with the dual norm, which is the operator norm when one recalls that $\mathbf{E}' = L(\mathbf{E}; \mathbf{C})$]

$\langle \mathbf{e}^*, \mathbf{e} \rangle$ or $\langle \mathbf{e}, \mathbf{e}^* \rangle$ the *duality bracket* between $\mathbf{e} \in \mathbf{E}$ and $\mathbf{e}^* \in \mathbf{E}^*$

$\bar{\mathbf{E}}'$ *antidual* of the Banach space **E**, i.e., the space of continuous antilinear functionals on **E** (a mapping $u : \mathbf{E} \to \mathbf{F}$ is *antilinear* if $u(\lambda \mathbf{x} + \mu \mathbf{y}) = \bar{\lambda} u(\mathbf{x}) + \bar{\mu} u(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in \mathbf{E}$, $\forall \lambda, \mu \in \mathbf{C}$)

$\langle \mathbf{x}^*, \mathbf{x} \rangle^-$ the bracket of the antiduality between **E** and $\bar{\mathbf{E}}'$

Main Spaces of Functions and Distributions

$C^0(\Omega)$ space of complex-valued functions, defined and continuous in the open set Ω, equipped with the topology of uniform convergence on the compact subsets of Ω

$C^0(\bar{\Omega})$ space of complex continuous functions on the closure $\bar{\Omega}$ (supposed to be compact), equipped with the maximum norm

$C^m(\Omega)$ space of complex functions, defined and m times continuously differentiable in Ω, equipped with the topology of uniform convergence on every compact subset of Ω, of the functions and of each one of their derivatives of order $< m + 1$ ($m \in \mathbf{Z}_+$ or $m = +\infty$)

$\mathscr{B}^m(\Omega)$ subspace of $C^m(\Omega)$ consisting of the functions having all their derivatives of order $< m + 1$ bounded in the whole of Ω, equipped with the topology of uniform convergence over Ω of all these derivatives

$C^m(\bar{\Omega})$ subspace of $\mathscr{B}^m(\Omega)$ consisting of the functions all of whose derivatives of order $< m + 1$ can be extended as continuous functions to the closure $\bar{\Omega}$ (supposed to be compact) of Ω, equipped with the topology induced by $\mathscr{B}^m(\Omega)$

$C_c^m(\Omega)$ subspace of $C^m(\Omega)$ consisting of the functions having *compact* support; elements of $C_c^\infty(\Omega)$ are often referred to as *test functions* in Ω

$C_c^m(K)$ subspace of $C_c^m(\mathbf{R}^n)$ whose support is contained in the compact set K, equipped with the topology induced by $C^m(\mathbf{R}^n)$

$L^p(\Omega)$ Lebesgue space of *measurable* functions f such that the pth power of the absolute value $|f|$ is integrable over Ω $(1 \leq p < +\infty)$, equipped with the norm $\|f\|_{L^p(\Omega)} = (\int_\Omega |f(x)|^p\,dx)^{1/p}$ (actually f represents an equivalence class of functions equal almost everywhere)

$L^\infty(\Omega)$ Lebesgue space of (classes of) measurable functions f in Ω which are essentially bounded, equipped with the norm $\|f\|_{L^\infty(\Omega)}$, the *essential supremum* of f

$L^p_{\text{loc}}(\Omega)$ space of locally-L^p functions f in Ω [i.e., if K is any compact subset of Ω, the function f_K equal to f on K and to zero in $\Omega\backslash K$ belongs to $L^p(\Omega)$]

$\mathscr{D}'(\Omega)$ space of distributions in Ω

$\mathscr{E}'(\Omega)$ space of distributions with compact support in Ω

$H^{m,p}(\Omega)$ space of functions u in Ω such that $D^\alpha u \in L^p(\Omega)$ for all n-tuples $\alpha \in \mathbf{Z}_+$, $|\alpha| \leqq m$ (D^α denotes the distribution derivative); $H^{m,p}(\Omega)$ is the *Sobolev space*, equipped with the norm

$$\|u\|_{H^{m,p}(\Omega)} = \Big\{\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\Big\}^{1/p}$$

$H_0^{m,p}(\Omega)$ closure in $H^{m,p}(\Omega)$ of $C_c^\infty(\Omega)$

$H^{-m,p}(\Omega)$ space of distributions u in Ω which can be written as finite sums of derivatives of order $\leq m$ of functions belonging to $L^p(\Omega)$ $(m \in \mathbf{Z}_+)$

$\mathscr{S}(\mathbf{R}^n)$ or $\mathscr{S}$ space of C^∞ functions u in $\mathbf{R}^n$ such that, for any pair of nonnegative integers k, M,

$$p_{k,M}(u) = \sup_{x\in\mathbf{R}^n}\Big\{(1 + |x|^2)^k \sum_{|\alpha|\leq M} |D^\alpha u(x)|\Big\} < +\infty,$$

equipped with the topology defined by the seminorms $p_{k,M}$ ($\mathscr{S}$ is the space of C^∞ functions in $\mathbf{R}^n$ *rapidly decaying at infinity*)

$\mathscr{S}'(\mathbf{R}^n)$ or $\mathscr{S}'$ the dual of $\mathscr{S}$, also the space of *tempered distributions* in $\mathbf{R}^n$

$H^s(\mathbf{R}^n)$ or H^s the *Sobolev space of order* $s \in \mathbf{R}$ in $\mathbf{R}^n$, i.e., the space of tempered distributions u in $\mathbf{R}^n$ whose Fourier transform $\hat{u}$ is a measurable function such that

$$\|u\|_s = \left(\int_{\mathbf{R}_n} |\hat{u}(\xi)|^2(1 + |\xi|^2)^s \frac{d\xi}{(2\pi)^n}\right)^{1/2} < +\infty,$$

equipped with the Hilbert space structure defined by the norm $\|\ \ \|_s$

$H^s(K)$	subspace of H^s consisting of the elements having their support contained in the compact set K
$H^s_c(\Omega)$	space of distributions in Ω which belong to the space $H^s(K)$ for some choice of the compact subset K of Ω
$H^s_{loc}(\Omega)$	space of distributions u in Ω such that $\alpha u \in H^s$ for every $\alpha \in C_c^\infty(\Omega)$
$\mathscr{M}(\Omega)$	space of *Radon measures* in $\Omega[\mathscr{M}(\Omega)$ is the dual of $C_c^0(\Omega)]$
$\mathscr{D}'_+(\mathbf{R}^1)$ or $\mathscr{D}'_+$	space of distributions on the real line which vanish identically in the open negative half-line

Note: When $\Omega = \mathbf{R}^n$, $(\mathbf{R}^n)$ will often be omitted, e.g., as in C^∞, $\mathscr{D}'$, H^s, L^p, etc.

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CHAPTER I

The Basic Examples of Linear PDEs and Their Fundamental Solutions

I

The Basic Examples of Linear PDEs

The theory of linear PDEs stems from the intensive study of a few special equations, whose importance was recognized in the eighteenth and nineteenth centuries. These were the basic equations in mathematical physics (gravitation, electromagnetism, sound propagation, heat transfer, and quantum mechanics). After their introduction in applied mathematics, they were shown to play important roles in pure mathematics: For instance, the Laplace equation was first studied as the basic equation in the theory of Newton's potential and in electrostatics; later, suitably reinterpreted, it was used to study the geometry and topology of Riemannian manifolds. Similarly, the heat equation was studied by Fourier in the context of heat transfer. Later it was shown to be related to probability theory. One of the basic examples, which we describe below, does not seem to have originated in applications to physics: the Cauchy–Riemann operator, which is used to define analytic functions of a complex variable. But to my knowledge, all the remaining ones have their origin in applied mathematics. At any rate, the general theory of linear PDEs is an elaboration of the respective theories of these special operators. During the twentieth century it was recognized that many properties which had seemed to be the prerogative of the Laplace equation or of the wave equation could in fact be extended to wide classes of equations. These properties usually center around a question or a problem that only makes sense for one or the other equation: for instance, around the Dirichlet problem, which makes sense for the Laplace equation but not really for the wave equation, or the Cauchy problem, which is well posed for the latter but not for the former. The purpose of this introductory course is to help the student to understand some of these problems and some of their solutions—but always by staying very close to the special equation for which they were originally considered. It is therefore necessary that we have the nature of the basic examples clearly in mind.