Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

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V. Villani (Ed.)

Complex Geometry and Analysis

Proceedings, Pisa 1988

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in honour of Edoardo Vesentini
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Foreword

This volume contains the texts of the main talks delivered at the International Symposium on Complex Geometry and Analysis held in Pisa, May 23–27, 1988. The Symposium was organized on the occasion of the sixtieth birthday of Edoardo Vesentini, by some of his former students, in appreciation of his many contributions to mathematics, of his teaching and advice.

The aim of the lectures was to describe the present situation, the recent developments and research trends in several relevant topics in Complex Geometry and Analysis, that is in those fields in which the mathematical activity of E. Vesentini is most fruitful and inspiring. The contributors are distinguished mathematicians who have actively collaborated with the mathematical school in Pisa over the past thirty years.

The organizers would like to thank all the supporting institutions, and, in particular, the Comitato per la Matematica (CNR) and the Gruppo Nazionale di Geometria Analitica ed Analisi Complessa (MPI).

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HYPERKÄHLER MANIFOLDS

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§1. Introduction and Definitions

In recent years hyperkähler manifolds have turned up in a wide variety of contexts, and it is now becoming clear that they form a very interesting class of manifolds with a rich theory. The purpose of this lecture is to justify these claims by giving an overall survey of the field.

I shall begin by reviewing the basic definitions and elementary properties. Then in §2 I will describe the hyperkähler quotient construction of [7] which enables us to construct many examples quite painlessly. This shows that the theory has a definitely non-trivial content. In §3 I will concentrate on 4-dimensional manifolds which are of special interest for various reasons including the classical relation to physics via Einstein's equations. In particular, I will describe the beautiful family of examples due to P.B. Kronheimer [9]. In §4 I explain how Yang-Mills moduli spaces give yet more examples of hyperkähler manifolds. In particular the moduli spaces of magnetic monopoles studied in §2 are of special interest. Finally in §5 I describe the twistor theory of R. Penrose as it applies to hyperkähler manifolds and illustrate it for the case of monopole spaces.

As the name rather obviously suggests hyperkähler manifolds are a generalization of Kähler manifolds, so it is best to start by briefly recalling that a Kähler manifold may be defined as a Riemannian manifold X with an almost complex structure I (orthogonal transformation of the tangent bundle with $I^2 = -1$) which is covariant constant. This condition implies the usual integrability condition for I so that X is actually a complex manifold. Equivalently a Kähler manifold is a Riemannian manifold with holonomy group contained in

$U(n) \subset SO(2n)$.

The importance of Kähler manifolds lies mainly in the fact that algebraic manifolds (affine or projective) always carry Kähler metrics. It is also significant that the 2-form ω_I associated to I and the metric is closed and non-degenerate, so that Kähler manifolds are in particular symplectic.

Next let us recall that the algebra H of quaternions is generated (over R) by the symbols i, j, k with the relations

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \quad \text{etc.}$$

or more succinctly, for $x,y,z \in R$,

$$(xi + yj + zk)^2 = -(x^2 + y^2 + z^2) \qquad (1.1)$$

A hyperkähler manifold is now defined as a Riemannian manifold X endowed with I, J, K (orthogonal transformations of the tangent bundle) satisfying the quaternion algebra identities and covariant constant. Briefly we may say that the tangent spaces to X have a covariant constant H-module structure. Equivalently the holonomy group of X lies in the symplectic group

$Sp(k) \subset SO(4k)$.

Clearly by choosing the structure I and ignoring J, K we see that X has in particular a complex Kähler structure. More generally the role of I can be replaced by

$$I_\lambda = xI + yJ + zK$$

where

$$\lambda = (x,y,z) \in R^3 \quad \text{with} \quad \lambda^2 = x^2 + y^2 + z^2 = 1 .$$

This shows that X has a whole family of complex structures, parametrized by points λ of the 2-sphere, and that the metric is

Kählerian for all these complex structures. This explains the terminology "hyperkähler".

Note. Although logical and descriptive the terminology is rather cumbersome and a beautiful class of manifolds deserves a better fate. Because they involve so many of Hamilton's main interests (quaternions, symplectic geometry, theoretical physics) I proposed at one stage that they should be christened "Hamiltonian manifolds". Unfortunately the hyperkähler usage was too widely used (expecially by physicists) to be eradicated. A pity!

The 3 operators I, J, K combined with the metric yield 3 covariant constant 2-forms ω_I, ω_J, ω_K giving 3 symplectic structures. If we fix on the complex structure defined by I then ω_I is the (1,1) form associated to the Kähler metric, while $\omega_J + i\omega_K$ becomes a closed holomorphic 2-form defining a "holomorphic symplectic" structure.

Since Kähler manifolds play an important role in complex algebraic geometry one might speculate that hyperkähler manifolds should play a similarly important role in "quaternionic algebraic geometry". Unfortunately quaternionic algebraic geometry does not seem to exist. In fact, as we shall see, one can argue in reverse that hyperkähler geometry provides a substitute for the non-existent quaternionic algebraic geometry. The arguments for this view-point are strengthened by the following important fact: an irreducible hyperkähler metric is uniquely determined (up to a constant scale factor) by its family of complex structures. Note that the corresponding result for Kähler metrics is totally false: there are many Kähler metrics on a fixed complex manifold. In this sense hyperkähler geometry is more tightly related to complex analysis (and eventually to algebra) than Kähler geometry. This becomes clearer in the twistor picture which we shall explain in §5.

Clearly quaternionic space H^k with standard metric (where i, j, k act orthogonally) is a hyperkähler manifold. These linear or flat examples are not very interesting, but they provide the starting point for the construction of non-linear examples as we shall see in the next section.

§2. The quotient construction

It will be clear from the definitions in §1 that a hyperkähler structure is a very restricted one, and one might tend to dismiss the theory as having only a mild specialized interest. That was certainly my initial reaction, but my view was radically changed by the discovery in [7] of a very simple and beautiful "quotient construction" which generates vast numbers of hyperkähler manifolds in a natural way. Moreover this quotient construction is the quaternionic analogue of a Kähler quotient which is the geometric version of classical invariant theory. In this sense the hyperkähler quotient replaces the non-existent "quaternionic invariant theory".

Let me begin therefore by reviewing the quotient construction in Kähler geometry. The prototype is provided by considering the standard action (scalar multiplication) of the circle group S^1 on a complex vector space C^n. The standard way in algebraic geometry to form a quotient is to complexify S^1 to the complex multiplicative group C^*, then remove the origin from C^n (a "bad" point) and to form the projective space

$$P_{n-1} = (C^n - 0)/C^* . \qquad (2.1)$$

An equivalent procedure using real differential geometry is to restrict the action of S^1 to the unit sphere S^{2n-1}, so that

$$P_{n-1} = S^{2n-1}/S^1 . \qquad (2.2)$$

In this guise P_{n-1} inherits a natural metric, but the complex structure is not so transparant. The link between the complex and metric view-points lies in symplectic geometry. In fact the function $|z|^2$ on C^n viewed as a Hamiltonian, with respect to the symplectic structure of C^n given by its standard hermitian metric, generates the Hamiltonian flow of the S^1-action. The quotient (2.2) inherits a natural symplectic structure, a procedure well known in classicial mechanics.

This simple example generalizes to the action of any compact (connected) Lie group G on a Kähler manifold X. We assume that G preserves both the metric and the complex structure, hence also the symplectic structure. Under mild conditions there is then a moment map

$$\mu : X \to g^* \qquad (2.3)$$

where g^* is the dual of the Lie algebra of G. The components of μ are Hamiltonian functions corresponding to the flows defined by one-parameter subgroups of G. Also μ is assumed to be G-equivariant. Now let $\alpha \in g^*$ be fixed by G (frequently we take $\alpha = 0$) and assume it is a regular value for μ. Then the manifold

$$X_\alpha = \mu^{-1}(\alpha)/G$$

inherits a natural symplectic structure. Clearly X_α also inherits a Riemannian metric. Together with the symplectic form ω this then defines an almost complex structure I which makes X_α a Kähler manifold, the Kähler quotient [8].

If X is a projective algebraic variety (with Kähler class coming from a projective embedding) then X_α is the projective variety whose coordinate ring is essentially the G-invariant part of the coordinate ring of X. All this is part of "geometric invariant theory" as developed by Mumford.

We are now ready for the hyperkähler case, so let X be a hyperkähler manifold and let G be a compact Lie group of automorphisms of X. Using the 3 symplectic structures ω_I, ω_J, ω_K of X we get (under mild assumptions) 3 moment maps μ_I, μ_J, μ_K which we can combine into a single quaternionic moment map

$$\mu : X \to g^* \otimes R^3$$

which is G-equivariant. Let $\alpha \in g^* \otimes R^3$ be fixed by G and assume this is a regular value of μ. Then the manifold

$$X_\alpha = \mu^{-1}(\alpha)/G$$

has 3 induced symplectic structures which, together with the induced metric, define a hyperkähler structure. This is the hyperkähler quotient of [7].

The complex structure I of X_α can be seen from an alternative description. Recall that $\omega_J + i\omega_K$ defines a holomorphic symplectic structure on X. The holomorphic action of G^c preserves this and $\mu_J + i\mu_K$ gives a holomorphic moment map μ^c. Then $(\mu^c)^{-1}(\alpha_J + i\alpha_K)$ is a complex submanifold Y of X acted on by G and X_α is clearly the same as the Kähler quotient Y_{α_I}, where $\alpha_I, \alpha_J, \alpha_K \in g^*$ are the 3 components of $\alpha \in g^* \otimes R^3$.

Notes. 1) As the title of [7] indicates hyperkähler manifolds are of interest to physicists in relation to supersymmetric models.

2) If X is complete the hyperkähler quotient X_α is also complete. If α is not a regular value of μ then X_α will have singularities and removing these leads to an incomplete manifold.

As I pointed out in §1 the quaternionic spaces H^k are hyperkähler manifolds. Hence if $G \to Sp(k) = Aut(H^k)$ is any symplectic representation of G we are in the situation where we can try to construct quotient hyperkähler manifolds from the action of G on H^k. Since there are many choices of groups and representation we see that the quotient construction will lead to very many hyperkähler manifolds. Even when G is a circle or torus the construction yields interesting examples.

§3. 4-dimensional examples

Since a hyperkähler manifold has dimension $4k$ the lowest dimension is 4, i.e. quaternionic dimension 1. These are in a sense the quaternionic analogues of Riemann surfaces or algebraic curves and, as such, deserve special attention. They are also of special interest because 4 is the dimension of space-time and, since $Sp(1) = SU(2)$, a hyperkähler 4-manifold is the same as a Kähler-Einstein (or self-dual Einstein) manifold. As solutions of the (positive definite) Einstein equations such manifolds have been studied by pysicists in connection with the quantization of gravity. They are referred to as "gravitational instantons".

So far I have not discussed questions of compactness or completeness but these are obviously important aspects. Compact 4-dimensional examples are scarce and essentially consist of flat tori and the K3 surfaces where the existence of a Kähler-Einstein metric has been established by S.T. Yau with his proof of the Calabi conjecture.

If we consider non-compact manifolds the next simplest class would be complete manifolds which are asymptotically flat. In fact this can be interpreted in a number of slightly different ways. One class (referred to as ALE spaces: asymptotically locally Euclidean) requires the manifold to behave at ∞ like $(R^4-0)/\Gamma$ where $\Gamma \subset Sp(1)$ is a finite group.

Since $Sp(1) = SU(2)$ double covers $SO(3)$ the groups Γ which can occur are just the double covers of the symmetry groups of the Platonic regular solids in R^3, namely the cyclic, dihedral, tetrahedral, octahedral and icosahedral groups. These groups are well-known to be linked, in a subtle way, to the simply-laced Lie groups A_n, D_n, E_6, E_7, E_8 .

The construction and classification of ALE spaces for all choices of Γ has been worked out by P.B. Kronheimer [9] in a very beautiful theory. He constructs his manifolds as hyperkähler quotients with a judicious choice of Lie group G and symplectic representation. These are determined uniformly, for all Γ, in terms of the representation theory of Γ, the key ingredients being the regular representation and the 2-dimensional representation $\Gamma \to Sp(1) = SU(2)$ from which Γ arose. Moreover, the hyperkähler metrics have moduli which arise from the choice of α for the value of the moment map. Kronheimer proves that the moduli space can be naturally identified with an open set of "regular" points in the quotient

$$(h \otimes R^3)/W$$

where h is the Cartan algebra of the corresponding Lie group and W is its Weyl group.

If we consider these 4-dimensional hyperkähler manifolds as "quaternionic algebraic curves" they are analogous in many respects to complex algebraic curves. H^1 of complex curves is replaced by H^2 of our "quaternionic curves" so that

$$\text{rank } H^2 \ (= \dim h)$$

is analogous to the genus. The most direct analogy would restrict us to the cyclic groups (type A_n) but the quaternionic case is richer since we have another infinite family (D_n) and the 3 exceptional cases. Also the moduli are determined by period matrices in all cases: we integrate the 3 covariant constant 2-forms over a basis of H_2.

The A_n family were previously known due to work of Eguchi-Hanson, Gibbons-Hawking and Hitchin. Also Kronheimer's work has an intimate relation with that of Brieskorn[5] on deformations and resolutions of rational double points.

§4. Yang-Mills moduli spaces

If we accept that hyperkähler 4-manifolds are like algebraic curves then we might conjecture that it should be possible to construct higher dimensional examples as moduli spaces for bundles over "curves". This turns out to be true as I shall now explain.

Let X be a hyperkähler 4-manifold, let G be a compact Lie group and let A be the space of all G-connections for a fixed G-bundle P over X. Then A is an affine space modelled on 1-forms on X with values in g. The I, J, K operators induce similar operators on A which makes A an ∞-dimensional affine space over H, with a compatible metric. Moreover the gauge group $G = \mathrm{Aut}(P)$ acts naturally on A preserving its affine, metric and quaternionic structures. We can therefore consider (rather formally) the ∞-dimensional hyperkähler moment map

$$\mu : A \to (\mathrm{Lie}\ G)^* \otimes R^3$$

and then try to construct hyperkähler quotients.

In fact a little computation (with appropriate care being taken over the non-compactness of X) shows that μ is essentially the self-dual part of the curvature. Thus $\mu = 0$ becomes the <u>(anti)-self-dual Yang-Mills</u> equations which define <u>instantons</u> on X, and the hyperkähler quotient

$$M = \mu^{-1}(0)/G$$

is just the instanton moduli space so extensively studied in general by Donaldson.

There are various cases of special interest, of which the simplest arise for $X = R^4$ or $S^1 \times R^3$. The first gives the instanton moduli spaces studied in [1], while the S^1-invariant part of the moduli space for $S^1 \times R^3$ gives the magnetic monopole moduli spaces studied in [2].

Of course this description is very formal and ignores the analytical difficulties that arise with ∞-dimensional spaces. Nevertheless the analysis works and the conclusions remain valid so that we have here families of hyperkähler manifolds, which arise naturally as hyperkähler quotients of ∞-dimensional affine spaces.

There is actually a very mysterious duality principle of Nahm which means that the same moduli space has 2 different (dual) presentations as a hyperkähler quotient. Roughly speaking Nahm's principle goes as follows. Let $\Gamma \subset R^4$ be a subgroup of the form $R^a \times Z^b$ and let Γ' be the Pontrjagin dual (or character group) of R^4/Γ. Then we can construct Γ-invariant instantons from Γ'-invariant instantons and vice-versa. However, the Lie groups involved are not the same, the duality interchanging quantities like the rank of the Lie group and Chern classes.

In the extreme case when $\Gamma = 0$, then $\Gamma' = R^4$ and Γ'-invariance reduces us to algebra. In fact Nahm's principle, as shown by Donaldson [6], amounts to the main result in [1], [3] which gives an algebraic description of the instanton moduli space. In this case therefore the instanton moduli space has 2 hyperkähler quotient descriptions, one finite-dimensional described in [6] and the other ∞-dimensional.

When $\Gamma = R$ we are in the case originally studied by Nahm and leading to the monopole moduli spaces of [2].

The hyperkähler metric on the instanton moduli spaces of R^4 is, for rather basic reasons, incomplete. For example the first case is $H \times (H-0)/Z_2$ with the flat metric. On the other hand the monopole moduli spaces have complete metrics and this completeness has an important physical interpretation as explained in [2].

These monopole spaces are therefore an interesting class of hyperkähler manifolds and I will return to them in §5. Let me at this stage just say that they have somewhat different asymptotic properties to the ALE spaces.

§5. Twistor Spaces

Twistor spaces were introduced by R. Penrose into theoretical physics with the aim of translating problems from Minkowski space into an alternative framework where complex analysis and geometry can be brought into play. Hyperkähler manifolds fit naturally into the Penrose twistor theory. In fact the 4-dimensional case involving Einstein's equation represents part of the motivation and also the success of the Penrose programme.

The basic idea is very simple. Since a hyperkähler manifold X has a family of complex structures I_λ parametrized by $\lambda \in S^2 = P_1(C)$ we can put all these together on $X \times P_1$. If we put the complex structure I_λ on the fibre $X_\lambda = X \times \{\lambda\}$ and give P_1 its natural complex structure it turns out that we get a complex structure on the total space $Z = X \times P_1$ so that the projection $Z \to P_1$ is a holomorphic map. In terms of the general theory of complex structures we can say that the X_λ form a holomorphic family of complex structures. Z is called the <u>twistor space</u>.

If σ is the antipodal map on $S^2 = P_1$ then $X_{\sigma(\lambda)}$ is the complex conjugate structure to X_λ. Thus $(x,\lambda) \to (x,\sigma(\lambda))$ extends σ to a complex conjugation or real structure on Z. The horizontal sections $\{x\} \times P_1$ are holomorphic curves and are real (i.e. σ-invariant).

By adding a bit more data, essentially the holomorphic symplectic structures on the fibres X_λ we end up with a twistor description (involving only holomorphic data and the real structure σ) which is entirely equivalent to the hyperkähler metric of X.

Returning to our general idea that hyperkähler manifolds provide a substitute for quaternion algebraic varieties the twistor philosophy can now be summarized as follows. Instead of trying to develop a theory of non-commtative quaternionic analysis we use ordinary complex analysis for all embeddings $C \to H$, parametrized by $\lambda \in P_1$, and remember also the holomorphic dependence on λ.

The twistor picture suggests new ways of generating hyperkähler manifolds. For example given a twistor space $Z \to P_1$ we could try to replace each fibre X_λ by a suitable desingularization $X_\lambda(k)$ of its k-fold symmetric product so as to obtain a new twistor space $Z(k) \to P_1$. For this to work the new fibres $X_\lambda(k)$ have to be holomorphic symplectic manifolds. This procedure turns out to work when $\dim X = 4$, so that the X_λ are complex surfaces. The desingularization needed uses Hilbert schemes as in the work of Beauville [4].

Consider in particular the case $X = S^1 \times R^3$ with its standard flat metric. In terms of the magnetic monopoles (for SU(2)) studied in [2] we can identify X with the moduli space M_1 of 1-monopoles: such a monopole has a "location" in R^3 and a "phase" angle. The k-monopole moduli space M_k is, as we observed earlier, a hyperkähler manifold. Its twistor space $Z(k)$ is obtained from the twistor space Z of $M_1 = S^1 \times R^3$ by a version of the desingularized k-fold symmetric product construction indicated above. This means that the horizontal sections of $Z(k) \to P_1$ (which represent points of M_k and hence k-monopoles) correspond to k-sections of $Z \to P_1$, i.e. holomorphic curves meeting each fibre in k points (possibly coincident).

This representation of k-monopoles in R^3 by k-sections of the twistor space is intimately related to "soliton" ideas. I recall that a 1-monopole is viewed as an approximately localized magnetic particle, and a k-soliton can be viewed approximately as a superposition of k such particles provided these are far apart. However when the particles get close together the k-monopole loses its particle identity and is just a complicated non-linear field in space. Translated into the twistor picture this says that a k-monopole, in the far separated case, is represented by a k-section which approximately looks like a union of k simple sections. In general however a k-section does not resemble k separate sections.

The twistor picture enables us to take the soliton idea one stage further. If we fix one fibre of $Z \to P_1$, i.e. if we fix a complex structure of $S^1 \times R^3$, then a k-section does indeed cut this fibre in just k points (possibly coincident), and these determine the k-section. We can in this way think of the general k-monopole as an exact "superposition" of k single monopoles. This description depends however on the choice of complex structure on $S^1 \times R^3$. The dependence is weak in the far separated case (so that we recover the usual soliton picture) but is strong in the nearby (or interactive) case.

Since solitons are one version of the particle/wave dichotomy I like to think that Hamilton, who was much involved in the 19th century controversies on the nature of light, would have been intrigued by the role which quaternions play in connection with solitons.

Let me conclude with a few brief remarks about the first non-trivial monopole space, namely the 2-monopole moduli space M_2 . Because there is a natural centre it turns out that, up to a double covering, M_2 is the product of M_1 (representing the centre of mass) and another hyperkähler 4-manifold M_2^o which measures variables relative to the centre. The manifold M_2^o is a very remarkable 4-dimensional hyperkähler manifold and it is extensively studied in [2]. Here are some of its basic properties.

(1) Asymptotically it looks like a circle bundle over $R^3 - 0$,

(2) The fundamental group at ∞ is the quaternion group of order 8,

(3) It admits an action of SO(3) by isometries; this action does not preserve the complex structures, but rotates them,

(4) Its fundamental group is of order 2 and its double covering is the algebraic surface

$$x^2 - zy^2 = 1$$

Property (3) and the hyperkähler property essentially determine the metric uniquely and there is an explicit formula for it involving elliptic integrals. Except for an overall scale there are no free parameters. The geodesics on M_2^o have an interpretation in terms of the dynamics of slowly moving monopoles and this is the main theorem of [2].