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Edited by A. Dold, B. Eckmann and F. Takens

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M. Mimura (Ed.)

Homotopy Theory and Related Topics

Proceedings, Kinosaki 1988

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Proceedings of the International Conference held at Kinosaki, Japan, August 19–24, 1988

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Preface.

An international conference on HOMOTOPY THEORY AND RELATED TOPICS was held August 19-24, 1988 at the Kinosaki Conference Hall in Hyogo, Japan in honor of Professor Hirosi Toda who celebrated his sixtieth birthday on January 17, 1988. His influence on the development of homotopy theory has been enormous.

The organizing committee consisted of

A. Kono, Kyoto University

M. Mahowald, Northwestern University

M. Mimura, Okayama University

K. Morisugi, Wakayama University

G. Nishida, Kyoto University

F.P. Peterson, Massachusetts Institute of Technology

The titles of the talks delivered at the conference are given below. The proceedings contain transcripts of some of the conference talks as well as related articles. All papers were refereed, and we take this opportunity to thank the referees. We would also like to express our gratitude to the participants for their contributions to the conference and particularly to K. Iriye for help with various arrangements.

Okayama, JAPAN

September 1989

M. Mimura

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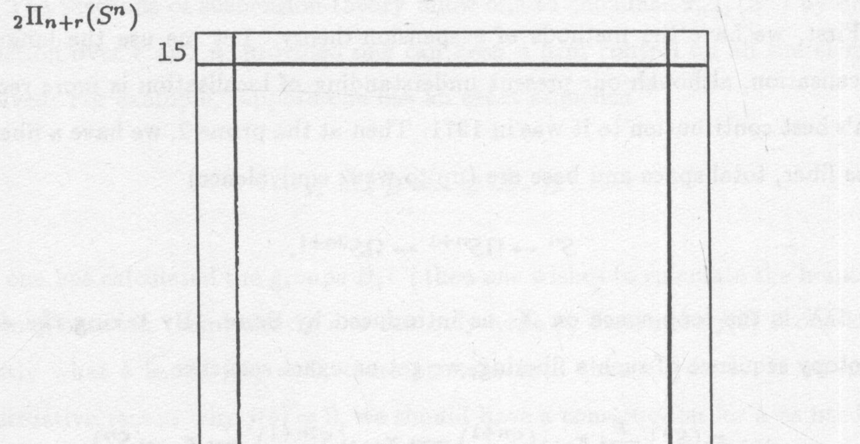
Talk on Toda's work

J. F. Adams

I thank you for the privilege of giving this lecture. Prof. Toda has published about 70 papers on topology. It would seem to follow that some of these papers will get less than a minute of our time each. In fact, I can hardly attempt an exhaustive presentation of Toda's work. It will be better for all of us if I concentrate on a few key points. Let me begin by taking you back in time.

I first met Prof. Toda in 1955, at a meeting in Oxford which Henry Whitehead had organised and called the Young Topologists' Conference. You will quickly calculate that both Prof. Toda and I were younger then. Toda had spent the previous year in France and his English was not so fluent as it became later. Still, he could write on the blackboard. He wrote on the blackboard a large table.



And he proceeded to fill the table in, beginning with the groups we already knew, and proceeding to those we didn't know. Where there was something noteworthy he pointed to it, turned to the audience and smiled politely; and when he got to a group Z_{2^5} the audience clapped a little, for this was a new record for 2-torsion. Finally he indicated the Hopf invariant $H : \pi_{31}(S^{16}) \to \mathbb{Z}$. He wasn't the only person interested in that invariant, because Hopf was sitting in the front row. Hopf wasn't a young topologist even then, but Henry had made a few exceptions. Toda communicated that

$$H^{-1}\{1\} = \emptyset.$$

Hopf became most interested. He wanted to know the status of this assertion. Was this a theorem Toda had proved, or was it a conjecture or the hypothesis of something to follow?

John Moore, who was sitting beside Hopf, tried to reassure him. "That's solid." Unfortunately Hopf understood British English but not American English, so further interpreters were required. But in the end, everyone was happy with this theorem.

The methods by which Toda had done these calculations, and proved this result, were basically those set out in his book, "Composition Methods in Homotopy Groups of Spheres," Princeton Univ. Press 1962.

First, we have the methods of suspension-theory. Let me use the language of localisation, although our present understanding of localisation is more recent. Toda's best contribution to it was in 1971. Then at the prime 2, we have a fibering whose fiber, total space and base are (up to weak equivalence)

$$S^n \rightarrow \Omega S^{n+1} \rightarrow \Omega S^{2n+1}.$$

Here ΩX is the loop-space on X, as introduced by Serre. By taking the exact homotopy sequence of such a fibering, we get an exact sequence

$$\cdots \pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1}) \longrightarrow \pi_{r+1}(S^{2n+1}) \longrightarrow \pi_{r-1}(S^n)$$

in which the first arrow is the Freudenthal suspension E. So much is due to James.

At an odd prime p, matters depend on the parity of n. For the even case we have

$$\Omega S^{2m} \simeq S^{2m-1} \times \Omega S^{4m-1}$$

so the case of an even-dimensional sphere is reduced to the case of odd spheres. We now wish to study the double suspension

$$E^2 : \pi_r(S^{2n-1}) \rightarrow \pi_{r+2}(S^{2n+1}).$$

Toda studied it in two steps: first, at p there is a fibering

$$X \to \Omega S^{2n+1} \to \Omega S^{2pn+1}$$

where $X = S^{2n} \cup e^{4n} \cup e^{6n} \cup \cdots \cup e^{2(p-1)n}$ is the first part of a CW-model for ΩS^{2n+1}. Here I should point out that Toda had possessed a CW-model for the loops on a suspension ΩSX since 1953, which is two years before Jamess' work appeared in print. At all events, there is at p a fibering

$$S^{2n-1} \to \Omega X \to \Omega S^{2pn-1}.$$

It seems fair to say that Toda's insight into such function-spaces and the fiberings in which they take part remained unequalled for twenty years. The work of Cohen, Moore and Neisendorfer didn't start to appear till 1979.

The methods of suspension-theory allow one to calculate $\pi_{n+r}(S^n)$ by double induction over r and n, provided one can keep a firm control on all the elements involved. For example, suppose one has an exact sequence

$$\cdots A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} D \cdots$$

and one has calculated the groups B, C; then one wishes to calculate the homomorphism j between them; clearly one can only hope to determine $j(b)$ if one knows exactly what b is. If $j(b) = 0$, there exists a such that $i(a) = b$. If we have a constructive reason why $j(b) = 0$, we should have a construction for a as needed.

So secondly, we have methods of explicit geometric construction. In the homotopy groups of spheres we have the following primary operations: addition and subtraction; composition; Whitehead product. It was proved by Hilton that these exhaust the primary operations.

In fact, these operations suffice to explain the main point of Toda's proof about $\pi_{31}(S^{16})$. Barratt and Hilton had already proved that, in the stable homotopy groups of spheres, composition is anticommutative. More precisely, suppose given maps

$$S^{a+b} \xrightarrow{f} S^b, \quad S^{c+d} \xrightarrow{g} S^d.$$

Then you can form the smash product

$$S^{a+b} \wedge S^{c+d} \xrightarrow{f\wedge g} S^b \wedge S^d$$

and write it in the two equivalent ways

$$(1 \wedge g)(f \wedge 1) = (f \wedge 1)(1 \wedge g)$$

or

$$(E^b g)(E^{c+d} f) = (-1)^{ac}(E^d f)(E^{a+b} g).$$

Suppose you suspend once less, and form

$$(E^{b-1} g)(E^{c+d-1} f) - (-1)^{ac}(E^{d-1} f)(E^{a+b-1} g).$$

Then you have something which vanishes if you suspend it once more. There is a reason for this; you can write it in terms of the Whitehead product. Also the answer is zero if either f or g is a suspension, so you expect the result to depend only on the Hopf invariants of f and g, and that's what you find:

$$= \pm[\iota_{b+d-1}, \iota_{b+d-1}] E^{\cdots} Hf \; E^{\cdots} Hg.$$

It's a theorem of Toda that this equality holds (under dimensional restrictions removed by Barratt). In particular, in $\pi_{29}(S^{15})$ we get

$$[\iota_{15}, \iota_{15}] = 2\sigma_{15}\,\sigma_{22} \neq 0,$$

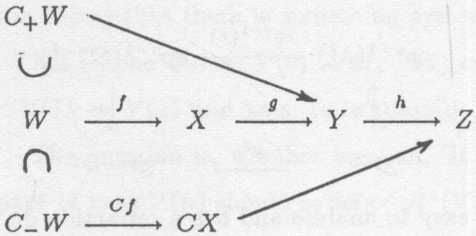
proving the result about the Hopf invariant on $\pi_{31}(S^{16})$. Of course, the same formula is used to identify other Whitehead products you need for $\pi_{n+14}(S^n)$, $n = 2, \ldots, 15$.

Still, when primary operations are not enough you have to turn to secondary ones. The most celebrated secondary operation in homotopy theory is one which

Toda introduced; he tried to call it the toric constructon, but everyone else calls it the Toda bracket. Suppose given 4 spaces and 3 maps

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

and suppose $hg \simeq 0$, $gf \simeq 0$, so that we can extend these composites over cones:



Then we have $hgf = 0$ for two different reasons; we get a map $SW \to Z$ by regarding SW as the union of the positive and negative cones. Taking account of the choices, we get a double coset in

$$h[SW,Y] \backslash [SW,Z] / [SX,Z]Sf.$$

These operations have good properties, and are important in Toda's calculations, and indeed in everyone else's calculations. But I will put this aside for now.

If we add the method of lifting homotopy groups, then we have mentioned the most important ingredients in Toda's book, but we shall have to hurry to mention some of what came later.

Next, about extended powers. Let X be a space with base-point (e.g. a finite CW-complex) and let G be a subgroup of Σ_n (e.g. $Z_p \subset \Sigma_p$). Then we can form

$$ep(X) = D(X) = \frac{(EG) \times_G \bigwedge_1^n X}{(EG) \times_G \bigwedge_1^n pt}$$

If you have a preferred CW-model for EG (as with $C = Z_p$) you write

$$ep^r(X) = \frac{(EG)^r \times_G \bigwedge_1^n X}{(EG)^r \times_G \bigwedge_1^n pt.}$$

Toda used these extended powers to prove the following result among others. Let $y \in \pi_i^S$, $py = 0$. Then $\alpha_1 y^p = 0$. The proof was, in principle, simple. If $py = 0$, then y extends to a map

$$M = S^m \cup_p e^{m+1} \xrightarrow{f} S^{m-t}.$$

This yields

$$\begin{array}{ccc} ep^{p-1}(M) & \xrightarrow{ep^{p-1}(f)} & ep^{p-1}(S^{m-t}) \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle r} \\ S^{pm} & \xrightarrow{y^p} & S^{pm-pt} \end{array}$$

Now $ep^{p-1}(S^{m-t})$ is easy to analyse and has a retraction on S^{pm-pt}. So the composite is y^p. But we can also look at the complex $ep^{p-1}(M)$ and find that $i\alpha_1 = 0$ here: so $y^p\alpha_1 = 0$.

In this way, Toda obtained his second proof that $\alpha_1\beta_1^p = 0$, which settled a problem and allowed him to push his calculations of the p components of homotopy groups of spheres up to the range $(p^2 + 2p)2(p-1) - 3$.

It would seem that this proof influenced Nishida in his work on the nilpotence of the stable homotopy groups of spheres. For, after all, the number n of factors does not have to be the same as p; if one increases n one may hope to make $ep^n(M)$ more like an Eilenberg-MacLane space and so prove $iy = 0$, $y^n y = 0$ provided $py = 0$.

And, as we know, the nilpotence theorem of Nishida influenced the nilpotence conjecture of Ravenel, which was the starting-point for the work of Hopkins. But I am getting out of historical order.

It appears from our account of the Toda bracket that it factors in the form

$$SW \xrightarrow{F} Y \cup_g CX \xrightarrow{H} Z.$$

That is, you can replace Toda brackets by ordinary composites, but even if W, X, Y, Z are spheres you have to be willing to consider complexes more general than spheres. For example, let $M = S^n \cup_p e^{n+1}$, $p \geq 3$. Then there is a map

$$S^{2(p-1)}M \xrightarrow{A} M,$$