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commutative algebra

analytic methods

edited by
Richard N. Draper

COMMUTATIVE ALGEBRA

Analytical Methods

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PREFACE

This volume consists of papers contributed by participants in the National Science Foundation-Conference Board of the Mathematical Sciences Regional Research Conference in Mathematics held at George Mason University from August 6 to August 10, 1979.

The central feature of the conference was a series of ten lectures delivered by Professor Melvin Hochster entitled "Analytic Methods in Commutative Algebra." The purpose of his lectures was to expose to the community of commutative algebraists two areas where analytic methods have been having impact on commutative algebra recently. The first is Hodge Theory and the Grauert-Riemenschneider Vanishing Theorem, which is affecting the study of singularities. The second is a recent quantitative result of Skoda and Briancon concerning elements integral over an ideal where the analytic technique involves relative boundedness of functions. The contents of Professor Hochster's lectures are expected to appear in the CBMS Regional Conference Series in Pure Mathematics published by the American Mathematical Society.

The papers in the present volume demonstrate the wide range of interaction that occurs between algebra and analysis. Since the contributors are primarily algebraists, the interaction most often takes the form of applications of analytic techniques to algebra. Moreover, the techniques used in these papers are much more widely known and used than those described in Professor Hochster's lectures. They consist, in large part, of applying ideas of analysis to an algebraic setting by drawing the appropriate analogy. Throughout these papers one finds a whole circle of ideas from analysis that have become fundamental in the study of commutative algebra. Among these are the concepts of local ring (meromorphic functions regular at a point), ideal-adic filtration and completion (approximation, Taylor series, and rings of convergent power series), graded rings (tangent cones), derivations (vector fields), projective modules (vector bundles), and differentials and resolutions (differentials). Another area of interaction comes from the ties between geometry and analysis. Much of commutative algebra has its roots in the study of algebraic varieties over the complex numbers, a subject having a long history of interaction with the study of analytic functions. Examples of this kind of interaction also occur herein. Brought together in one volume, these papers attest to the breadth and vigor of commutative algebra today.

There are a number of people whom it is my pleasure to thank for their contribution to the conference. First and foremost among them is Melvin Hochster, without whom there would have been no conference. I know that I speak for the participants and the faculty at George Mason University as well as for myself when I thank him for undertaking to deliver a series of lectures on such a demanding topic. We fully expect his lecture notes to be as valuable to the community as those that grew out of his 1974 conference at the University of Nebraska. Thanks are also due to David Eisenbud, Henry Laufer, Joseph Lipman, and Paul Roberts, who gave hour-long talks complementing the content of Hochster's lectures.

I would like to thank Michael Gabel for acting as Assistant Director. His contributions were at least the equal of my own, and he deserves much of the credit for the conference's success. Klaus Fischer and John Oppelt deserve thanks for making special contributions to the conduct of the conference. I would like to thank the Conference Board of the Mathematical Sciences and the National Science Foundation and their respective representatives, Roger Wiegand and Alvin Thaler, for supporting the conference.

Special thanks are deserved by the staff of the Department of Mathematics at George Mason University, Norma Jean Hynes, Bette Evers, Elizabeth Evers, and Nanette Smith for their tireless assistance. All four of these people worked long and hard, often under adverse circumstances, in order to make the conference a success.

Finally I would like to thank Earl Taft, the editorial board of Marcel Dekker, and Herbert Snyder for encouraging the publication of this volume, the contributors for providing manuscripts so conscientiously, and Bette Evers for her many hours of work in typing this book.

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THE PICARD-SEVERI BASE NUMBER

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INTRODUCTION

The Picard base number or, equivalently, the Picard-Severi base number, was introduced by Picard about 1901 and a complete discussion of it can be found in the two volumes of Théorie des Fonctions Algébriques by Picard and Simart. We feel that any discussion of the number should commence with Picard's treatment. Accordingly, it is our aim to present the method used by Picard so that the reader may compare it with the algebraic approach employed by Severi.

First, let us suppose given an algebraic surface in projective 3-space, having only ordinary singularities, and described by $f(x, y, z) = 0$. A family of plane sections of this surface will be denoted by Hy, where y is the parameter. Also, suppose given a curve C on the surface where C may be realized by the intersection of $f(x, y, z) = 0$ with another surface. C will play the role of a log curve in what follows.

By choosing a generic plane section Hy, we can construct an Abelian integral of the third kind, I, which has residue +1 at each finite point of intersection of Hy with C and having residue -d at a point at infinity, where d is the order of C. Let Hy have genus p. Then I will have 2p cyclic

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periods, say $\omega_1, \ldots, \omega_{2p}$. If y is allowed to vary in some manner and then return to its initial value, the above periods may undergo a change, say $(\omega_1, \ldots, \omega_{2p}) \overset{S}{\rightarrow} (\omega_1', \ldots, \omega_{2p}')$. According to the one-dimensional topology of the surface, S may be described as follows:

$$
(S) \qquad \begin{array}{l}
\omega_1' = m_1^1\omega_1 + m_2^1\omega_2 + \ldots + m_{2p}^1\omega_{2p} + \mu^1 \\
\omega_2' = m_1^2\omega_1 + m_2^2\omega_2 + \ldots + m_{2p}^2\omega_{2p} + \mu^2 \\
\vdots \\
\omega_{2p}' = m_1^{2p}\omega_1 + m_2^{2p}\omega_2 + \ldots + m_{2p}^{2p}\omega_{2p} + \mu^{2p}
\end{array}
$$

where the m's and the μ's are integers. The m's are independent of C but the μ's are due to C. It is evident that the ω's are functions of y and, as is shown by Picard, any singularities of the ω's occur for those values of y for which the genus of the plane section is less than p. Such values of y are finite in number, say k, so that for a circulation of y about one of these values, one obtains a transformation of type (S). Next, Picard introduces 2p independent Abelian integrals of the 2nd kind, say $I_1, \ldots, I_{2p}$, attached to a generic Hy. He also forms the integrals $J_1, \ldots, J_\lambda$ of the type I, above, where $C_1, \ldots, C_\lambda$ are the respective log curves. Now, consider the Abelian integral of the 3^{rd} kind,

$$J = a_1I_1 + \ldots + a_{2p}I_{2p} + c_1J_1 + \ldots + c_\lambda J_\lambda$$

where the a's are functions of y and the c's are constants. Generally, the cyclic periods of J depend on y although the residues do not. One asks whether it is possible to determine the a's and c's so that the cyclic periods of J do not depend on y. This means that if the constants $K_1, \ldots, K_{2p}$ are the initial periods of J, then any transformation of type (S) would be given by

$$K_i = m_1^iK_1 + m_2^iK_2 + \ldots + m_{2p}^iK_{2p} + c_1\mu^i + c_2\nu^i + \ldots + c_\lambda\pi^i \qquad (\alpha)$$

where $i = 1, \ldots, 2p$ and the m's, c's, and $\mu^i, \nu^i, \ldots, \pi^i$ are all integers. Note that there would be a total of 2pk such relations with integer coefficients. The following theorem is proved by Picard:

THEOREM If the constants $K_1, \ldots, K_{2p}$ and $c_1, \ldots, c_\lambda$, not all zero, satisfy the 2pk relations of type (α), we can construct a Picard integral $\int R(x, y, z)dx + \int S(x, y, z)dy$ having the K's as its cyclic periods and the c's for its residues, the latter corresponding to the curves $C_1, \ldots, C_\lambda$, where the C's are prescribed in advance.

PROOF By hypothesis, the cyclic periods of J are $K_1, \ldots, K_{2p}$. We then have 2p equations to determine the a's given by

$$a_1\Omega_1^i + a_2\Omega_2^i + \ldots + a_{2p}\Omega_{2p}^i = K_i \qquad (\beta)$$

where $i = 1, \ldots, 2p$ and the Ω's are the periods of the $I_1, \ldots, I_{2p}$. For brevity we denote the system (β) by $E_i = K_i$, $i = 1, \ldots, 2p$. To see that the a's are rational functions of y, it is sufficient to show that the system (β) remains invariant when y describes any closed path and this amounts to choosing a path corresponding to a transformation of type (S), above. Such a transformation yields

$$m_1^iE_1 + m_2^iE_2 + \ldots + m_{2p}^iE_{2p} + c_1\mu^i + c_2\nu^i + \ldots + c_\lambda\pi^i = K_i,$$
$$i = 1, \ldots, 2p$$

and this system is equivalent to the system (β) due to the hypothesis that the K's and c's satisfy (α). Thus, the integral J relative to the curve H_y has its cyclic periods independent of y. We can write

$$J = \int R(x, \tilde{y}, z)dx$$

and we must find $S(x, y, z)$ so that $\int R(x, y, z)dx + S(x, y, z)dy$ has, with the possible exception of the curve at infinity, the curves $C_1, \ldots, C_\lambda$ as its only log curves with residues $c_1, \ldots, c_\lambda$. For the construction of S, Picard proceeds as follows: Let x_0 be arbitrarily chosen and designate by $z_1, \ldots, z_m$ the m roots of $f(x_0, y, z) = 0$, where m is the order of the surface. Consider

$$S = \frac{1}{m}\frac{\partial}{\partial y}\left[\int_{x_0, z_1}^{x, z} Rdx + \ldots + \int_{x_0, z_m}^{x, z} Rdx\right]$$

Since the integrals have periods independent of y, S has a unique value at each point of the surface. Further, S is a rational function of (x, y, z) since the integrals have no essential singularities. Finally, $\frac{\partial S}{\partial x} = \frac{1}{m}\frac{\partial}{\partial y}[mR] = \frac{\partial R}{\partial y}$ so that the integrability conditions are satisfied.

The Picard integral $\int$ Rdx + Sdy has only $C_1, \ldots, C_\lambda$ and possibly the curve at infinity as its log curves. [QED]

We point out that one can always satisfy the hypothesis of this theorem by taking λ large enough. This is then the crux of the Picard result. We may state with Picard the following: On the surface f(x, y, z) = 0, with ordinary singularities, one can find ρ irreducible, algebraic curves $C_1, \ldots, C_\rho$ such that there exists no Picard integral of the third kind having for its log curves all or any of the curves $C_1, \ldots, C_\rho$, only. However, there does exist such an integral having an arbitrary $(\rho + 1)^{st}$ curve of the surface together with all or part of the curves $C_1, \ldots, C_\lambda$ as its only log curves.

Algebraic Approach

In the rest of this article we will describe a proof of Severi's result that the number of algebraically independent curves on a nonsingular algebraic surface (over algebraically closed ground field of characteristic zero) is finite. The result was first published by Severi in [4]. Severi did not properly define the notion of algebraic dependence and his proof is vague. We will use the notion of algebraic dependence due to Chow and van der Waerden and describe a proof along the lines of Severi. A result of Picard, which is discussed in the introduction, is crucial to the proof. We have introduced the necessary notions and stated Picard's result in an algebraic form though we have not been able to give an algebraic proof.

NOTATION In all the discussion that follows, k denotes the ground field and it is assumed to be algebraically closed and of characteristic zero. The projective nonsingular surface on which we consider the algebraic systems of curves will be denoted by Φ. The word curve will mean either a virtual or effective curve. The symbol "$\equiv$" is used for linear equivalence and "=" is used for algebraic equivalence. If C and D are two curves then $(C \cdot D)$ and $(C \cdot D)_p$ will denote respectively the intersection number and the intersection number at a point P of C and D. We will denote the arithmetic genus of a curve C by the notation p(C).

We will use the properties of algebraic equivalence stated below. A detailed discussion and proper definitions can be found in [1] or [2]. If A, B, C, D, etc. are curves on a nonsingular surface Φ then

1. $A \equiv B \Rightarrow A = B$, i.e., linear equivalence implies algebraic equivalence.
2. Algebraic equivalence is an equivalence relation.
3. Effective curves of the same order belong to a finite number of algebraic systems and any two curves which belong to the same algebraic system are algebraically equivalent.
4. If $A = B$ then $(A \cdot C) = (B \cdot C)$ and if $A = 0$ then $(A \cdot C) = 0$ for any curve C.
5. If $A = B$ and $C = D$ then $A + B = C + D$.
6. If $A + C = B + C$ then $A = B$.

Differentials and the Picard's Result

In relation to the nonsingular surface Φ we have simple and double differentials. Let $\omega_1 = Mdx + Ndy$ and $\omega_2 = Rdxdy$ be typical simple and double differentials where M, N, and R are rational functions on Φ and $\{x, y\}$ is a separating transcendence basis for the function field of Φ. With regard to these differentials we have the following notions.

If P is a point on Φ, and x, y are uniformizing coordinates of P then we say that ω_1 is regular at P if M and N are regular at P and similarly ω_2 is regular at P if R is regular at P. A differential which is regular at every point of Φ is said to be a differential of the first kind or simply a regular differential. Let C be an irreducible curve of Φ. The regularity of the differentials at C is similarly defined by taking x and y to be uniformizing coordinates of C. If a differential is not regular at C then we say that C is a polar curve of that differential. Let Ord_C denote the order given by the valuation defined by C on Φ. We define

Order of ω_1 at C = $\mathrm{Min}\{\mathrm{ord}_C M, \mathrm{ord}_C N\}$
Order of ω_2 at C = $\mathrm{Ord}_C R$