ELECTROMAGNETIC THEORY AND GEOMETRICAL OPTICS

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VOLUME XII

Preface

During the academic year 1947–48 Rudolf K. Luneburg gave a course at New York University in which he presented his ideas on the relation of classical geometrical optics to electromagnetic theory and on the foundation of diffraction optics. He also introduced the notion of an asymptotic series solution of Maxwell's equation in which the geometrical optics field is the essence of the first term. It seemed desirable shortly after Luneburg's death in 1949 to give some more permanent and fuller form to his mimeographed lecture notes, and the first-named author made some efforts at that time to amplify and polish the notes and to prepare them for a book.

However, Luneburg's interest lay primarily in geometrical optics, and his concern in looking at asymptotic series solutions of Maxwell's equations and in obtaining an asymptotic evaluation of the integrals of diffraction optics was to show that what is neglected as the frequency of the source becomes infinite does not matter for geometrical optics. While at New York University, Luneburg was a member of what is now called the Division of Electromagnetic Research of the Courant Institute of Mathematical Sciences; the chief interest of this group has been to develop methodology for solving electromagnetic problems at frequencies well below the optical range. For this purpose the higher order terms in the asymptotic series considered by Luneburg are of great interest because they provide improvements over what geometrical optics supplies. Moreover, Luneburg's work suggested the idea of seeking asymptotic series solutions for a variety of initial and boundary value problems of electromagnetic theory. A number of men both within the Division and without then exploited this idea and tackled numerous mathematical problems encountered in the course of this work. It therefore seemed wiser to defer publication of Luneburg's original material until one could add an account of this significant extension. The pressure of numerous daily obligations prevented both authors from working at a rapid rate, and far too many years have passed from the time the original project was conceived.

Several other matters warrant attention. In the summer of 1944 Luneburg gave a course entitled *Mathematical Theory of Optics* at Brown

University. The lecture notes for this course were mimeographed, but nothing more was done with them for many years. The University of California Press has now undertaken to publish the notes in book form and this move is indeed welcome. There is relatively little overlap between that book and the present one. The presentation of geometrical optics for isotropic media and some material on rays and wave fronts in such media are roughly the same there as here, and there is just a mention of the notion of a series solution of Maxwell's equations in inverse powers of the frequency. But the bulk of the Brown University lectures is devoted to geometrical optics proper and to aberration theory. The latter material is not covered in the present book. On the other hand, the electromagnetic approach to geometrical optics for anisotropic media, the expansion of general time-dependent fields in power series and of time-harmonic fields in asymptotic series, the asymptotic expansion of the integrals of diffraction optics, and some typical applications of the entire body of material to electromagnetic problems are to be found only in the present work.

This book is addressed to physicists and engineers as well as to mathematicians, and so we have made the effort to give full mathematical details. Many of these may seem unnecessary to proficient mathematicians, but they may be helpful to others whose backgrounds and special talents lie in other domains. Insofar as rigor is concerned, some points could have been supplied but the need for them did not seem strong. Other points of rigor are not as yet established, and footnotes indicate the current state of these deficiencies.

Another matter is the form in which the material of this book is presented. Luneburg himself and those who followed up directly the ideas in his work were concerned with Maxwell's equations, but it became clear, through papers which generalized on results obtained for Maxwell's equations, that the proper mathematical domain for the theory of this book is the class of first order linear symmetric systems of hyperbolic partial differential equations. Most of the material then of this book could have been framed entirely in terms of such systems, and from the standpoint of the mathematician the material might have been more attractive. But those interested in electromagnetic theory would not have been happier (if at all happy about what they do find). To understand what the more general systems say for Maxwell's equations calls for quite a task of specialization and interpretation. Moreover, the extension to symmetric hyperbolic systems was made by generalizing the relatively concrete work originally done for Maxwell's equations. Such generalizations are a comparatively simple matter when the basic ideas are already at hand, and it is far better pedagogically to study the concrete case first.

The authors are indebted to many men and agencies. The first-named author enjoyed the support of a Fulbright lectureship and a John Simon Guggenheim Memorial Fellowship during a year in which a great deal of the final writing was done. Much of the research that appeared in the original publications which are the basis of the present work was supported by the Mathematics Division of the Air Force Office of Scientific Research and by the Air Force Cambridge Research Laboratories.

MORRIS KLINE
IRVIN W. KAY

November, 1964

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Introduction

I have a paper afloat, with an electromagnetic theory of light, which 'till I am convinced to the contrary, I hold to be great guns.

James Clerk Maxwell (1865)

1. The Objectives of This Book

We propose to explore and exploit the relationship between Maxwell's electromagnetic theory and geometrical optics. In view of the fact that Maxwell's theory supersedes the older geometrical optics the question arises: Why should we pursue this relationship? There are four major reasons for pursuing and clarifying it.

The first is the purely theoretical or academic problem of building a mathematical bridge between the two domains, electromagnetic theory and geometrical optics. The older bases (to be described and discussed below) for asserting that geometrical optics is a limiting case of electromagnetic theory are vague and from a mathematical standpoint highly unsatisfactory.

The second major reason for investigating the relationship in question is a practical one. To solve problems of electromagnetic theory, whether in the range of radio frequencies or visible light frequencies, we should solve Maxwell's equations with the appropriate initial and boundary conditions. However, as is well known, Maxwell's equations can be solved exactly for few problems. Hence physicists and engineers, especially those concerned with high frequency problems, have frequently resorted to the simpler methods of geometrical optics. Although these methods have proved remarkably efficacious in the optical domain, they are intrinsically limited; they do not furnish information about some of the most important phenomena such as diffraction, polarization, and interference, to say nothing about the numerical accuracy of what geometrical optics does yield. It is also a fact that optical research men are now looking more and more into diffraction effects, and these prime users of geometrical optics are entering into an electromagnetic treatment of optical problems. The new techniques for producing monochromatic light certainly make such a step all the more advisable. Hence the practical question becomes whether the establishment of a better link between Maxwell's theory and geometrical optics will also provide more useful approximate methods of solving electromagnetic problems. Insofar as high frequency problems are concerned, the answer, based on work of the years since about 1955, can already be given affirmatively. We shall present approximate methods of solving electromagnetic problems which improve on geometrical optics in several respects.

The third major reason for pursuing the relation of Maxwell's theory to geometrical optics is to build a better basis for diffraction optics. The integrals of diffraction optics are now being used freely in some electromagnetic diffraction problems and are the starting point of hosts of investigations in diffraction optics. We shall give a new formulation of optical diffraction problems (though we shall employ Kirchhoff's principle), and we shall derive from it a better approach to the diffraction optics of incoherent light, that is, light emanating from a collection of sources, e.g., a collection of atoms putting forth radiations which are random in phase and polarization.

This investigation serves a fourth purpose. The material of this book is concerned with the relationship between full time-dependent solutions of Maxwell's differential equations and a special related phenomenon, geometrical optics. In principle it is concerned with the relationship between a wave theory and a non-periodic phenomenon, the latter being in some sense a limiting case as a parameter (the wave length in electromagnetic theory) goes to zero. The concepts and methods to be treated carry over to broad classes of linear partial differential equations. Thus, in any branch of physics which rests on a single linear partial differential equation or a system of linear partial differential equations in the independent variables x, y, z, and t, corresponding to the exact time-dependent solutions whether general or time-harmonic, there may exist physically significant limiting solutions as some parameter goes to zero. For example, the same values which the material presented here may offer in relating electromagnetic theory and geometrical optics may be derived for the relationship between quantum mechanics and classical mechanics. The theory also suggests the construction of a geometrical acoustics as a limiting case of the linearized theory of acoustics, that is, the acoustics of waves of small amplitude. Similarly, waves in gases have limiting phenomena which are called shock waves, and these are the "geometrical optics" of the ordinary waves. We shall, in fact, see that the electromagnetic investigations to be surveyed in this book do indeed suggest new creations or new insights into other branches of physics.

2. Some Relevant History

To appreciate just what the problem of reconciling geometrical optics and electromagnetic theory amounts to we shall examine briefly the historical background.

Geometrical optics was created to explain the nature and behavior of light. Before the seventeenth century the knowledge of optics consisted of fragments. Euclid gave the law of reflection, and the Alexandrian Greeks concerned themselves with the phenomenon of refraction though they did not succeed in finding the precise law. The Greeks also knew that one could concentrate light by means of paraboloidal, spherical, and ellipsoidal reflectors. The Arabians and medieval Europeans continued the Greek efforts to obtain the law of refraction and performed experiments on light. But until 1600 the knowledge of light and geometrical optics was still fragmentary.

The science of geometrical optics was founded in the seventeenth century. René Descartes and Willebrord Snell discovered the law of refraction; Robert Boyle and Robert Hooke discovered interference; Olaf Römer established the finiteness of the velocity of light; F. M. Grimaldi and Hooke discovered diffraction; Erasmus Bartholinus discovered double refraction in Iceland spar; and Newton discovered dispersion.

Two physical theories of light were created in the seventeenth century. Christian Huygens formulated the geometrical "wave" theory of light,[1] and Newton formulated a mechanical theory of propagation of particles.[2] Huygens thought of light as a longitudinal motion of ether and as spreading out at a finite velocity from a point source. The farthermost position reached by the light in space filled out a surface which he called the front of the wave. In homogeneous media this surface is a sphere. To explain further how light propagates, Huygens supposed that, when the disturbance reached any point in the ether, this point imparted its motion

[1] C. Huygens: *Traité de la Lumière*, 1690. An English translation is available from the University of Chicago Press, Chicago, 1945, or Dover Publications, Inc., New York, 1962.

[2] I. Newton: *Opticks*, 1704. An English edition is available from Dover Publications, Inc., New York, 1952.

to all neighboring points. Thus, if the wave front (Fig. 1) at time t_1 were the surface S_1 and if P were a typical point on S_1, the point P communicated its motion to all points in its neighborhood and from P the light spread out in all directions. Its velocity in these various directions depended on the nature of the medium. Thus in some small interval of time (and in an isotropic medium) the front of the light emanating from a point would be a sphere with P as a center. The same would be true at any other point of the surface S_1, except that the radii of the spheres might differ as the medium differs along S_1. The new position of the front at some time t_2 greater than t_1 is the envelope in the mathematical sense of the family of spheres attached one to each point of S_1. (According to this theory there is also a backward wave. This backward wave troubled scientists until Kirchhoff showed under his formulation that it does not exist. We shall not pursue this historical point.) To explain reflection and refraction Huygens supposed that the same phenomenon takes place at each point on a reflecting or refracting interface when the front reaches it, except, of course, that no waves penetrate a totally reflecting surface.

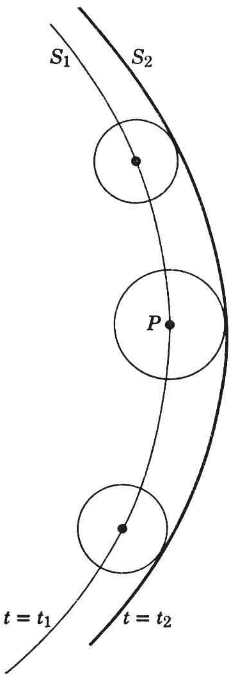

Figure 1

There are many more details to Huygens' theory which explain other phenomena of geometrical optics including double refraction. More relevant for us, however, is the fact that Huygens considered light as a series of successive impulses each traveling as already described, and he did not explain the relationship of the impulses to each other. Thus the periodicity of light is not contained in Huygens' theory. Also, although the phenomenon of diffraction had already been observed by Hooke and Grimaldi, Huygens apparently did not know it, and so his theory did not consider diffraction, although it could have covered at least a crude theory of this phenomenon.

The second major theory of light was Newton's. He suggested, in opposition to Huygens' "wave" theory, that a source of light emits a stream of particles in all directions in which the light propagates. These particles are distinct from the ether in which the particles move. In homogeneous space these particles travel in straight lines unless deflected by foreign bodies such as reflecting and refracting bodies. To account for the bright and dark rings which he observed when light passed through a plano-convex lens whose convex surface touched a plane surface Newton supposed that the plane surface reflected in "fits" which depended on the varying distance between the convex and plane surfaces. He admitted, however, that he did not know whether the light rays agitated the medium (ether), or the two surfaces bounding the air gap, or whether there was a vibrating or circulatory motion in the light stream itself. On the whole, Newton's theory was crude for the variety of phenomena he tried to embrace, and he made many *ad hoc* assumptions. Nevertheless Newton developed this mechanical theory so thoroughly that its completeness and Newton's own great reputation caused scientists, except for Euler and one or two others, to accept it for 100 years. Huygens' work was ignored.

Despite the recognition in the seventeenth century of phenomena such as diffraction, a limited theory of light, called geometrical optics, was erected on the basis of four principles. In homogeneous media light travels in straight lines. Light rays from a source travel out independently of one another. Light rays obey the law of reflection. And the rays obey the law of refraction at abrupt or discontinuous change in the medium. (The phenomenon of double refraction in crystals was also included by supposing that the medium has two indices of refraction which depend on position in space and the direction of propagation.)

All these laws follow from one embracing principle, Fermat's principle of least time. This principle presupposes that any medium is characterized by a function $n(x, y, z)$, the index of refraction (the absolute index or index relative to a vacuum). The optical distance between two points P_1 with coordinates (x_1, y_1, z_1) and P_2 with coordinates (x_2, y_2, z_2) over any given path is defined to be the line integral,

$$\int_{P_1}^{P_2} n(x, y, z)\, ds,$$

taken over that path. Fermat's principle, as stated by him and others following him, says that the optical path, the path which light actually takes, between P_1 and P_2, is that curve of all those joining P_1 and P_2 which makes the value of the integral least. This formulation is physically incorrect, as can be shown by examples, and the correct statement is that the first variation of this integral, in the sense of the calculus of variations, must be zero. This principle has been applied to the design of numerous optical instruments. It is to be noted that this principle or any other formulation of geometrical optics says nothing about the nature of light.

The mathematical theory of geometrical optics received its definitive formulation in the work of William R. Hamilton during the years 1824 to 1844.[3] Although Hamilton was aware of Fresnel's work, which we shall mention shortly, he ignored it. He was indifferent also to the physical interpretation, that is, Huygens' or Newton's, and to a possible extension to include interference. He was concerned with building a deductive, mathematical science of optics. He did include doubly refracting media (which are sometimes regarded as outside the pale of strict geometrical optics) and dispersion.

Hamilton's chief idea was the characteristic function, of which he gave several types. The basic one expresses the optical length of the ray which joins a point in the object space to a point in the image space as a function of the positions of these two points. The partial derivatives of this function give the direction of the light ray at the point in question. Of the three other types of characteristic functions which Hamilton introduced, one, the mixed characteristic, will be utilized in this book (Chapter XI). He showed that from a knowledge of any one of these functions all problems in optics, involving, for example, lenses, mirrors, crystals, and propagation in the atmosphere, can be solved. From Hamilton's work the equivalence of Fermat's principle and Huygens' principle is clear.

Incidentally, two of the characteristic functions which Hamilton introduced, the functions he designated by W and T, were rediscovered by Bruns (1848–1919) independently, who gave them the name eikonal.[4]

As we have already observed, geometrical optics cannot be regarded as an adequate theory of light because it does not take into account interference, diffraction, polarization, or even a measure of the intensity of light. In the early part of the nineteenth century new experimental work by Thomas Young, Augustin Fresnel, E. L. Malus, D. F. J. Arago, J. B. Biot, D. Brewster, W. H. Wollaston, and others made it clear that a wave theory of light was needed to account for all these phenomena. Fresnel extended Huygens' theory by adding periodicity in space and time to Huygens' wave fronts. Thereby interference was incorporated, and Fresnel also used the extended theory to explain diffraction as the mutual interference of the secondary waves emitted by those portions of the original wave front which are not obstructed by the diffracting obstacle.

Up to this time (1818) thinking on the wave theory of light (and for that matter even the corpuscular theory) had been guided by the analogy

[3] J. L. Synge and W. Conway: *The Mathematical Papers of Wm. R. Hamilton*, Vol. I, Cambridge University Press, London, 1931.

[4] H. Bruns: Das Eikonal, *Abhandl. Math.-Phys. Kl. Sächs. Akad. Wiss.*, **21**, 1895, 323–436.