HIGH SPEED COMPUTING OF ELASTIC STRUCTURES

TOME 2

LES CONGRÈS ET COLLOQUES DE L'UNIVERSITÉ DE LIÈGE

Volume 61

HIGH SPEED COMPUTING OF ELASTIC STRUCTURES

Proceedings of the Symposium of international Union of Theoretical and Applied Mechanics, held in Liège from August 23-28, 1970

Editor : B. FRAEIJS de VEUBEKE

UNIVERSITÉ DE LIÈGE
1971

CONTENTS

TOME 1

TOME 2

ON FURTHER APPLICATION OF THE FINITE ELEMENT METHOD TO THREE-DIMENSIONAL ELASTIC ANALYSIS

Y.R. RASHID *, P.D. SMITH **, N.PRINCE ***

1. Introduction

Departure from the simple constant-strain tetrahedral element (*1*) in favor of higher order elements (*2*) occurred at a relatively early stage in three-dimensional analysis of elastic structures, and complex elements of various shapes have been developed in recent years (*3-8*). The displacement approximations adopted for these complex elements include complete second and third order polynomials and various combinations of interpolation functions. If a simple criterion for selecting a superior element can be defined in terms of, first, its degree of accuracy and, second, its adaptability to irregular boundaries, then we clearly already have at our disposal many such elements which satisfy this criterion quite satisfactorily. It would seem, then, that any further development of basic element formulation is not likely to produce substantially better alternatives to existing element models, and the point of diminishing returns may have already been reached.

The three-dimensional computational problem, however, goes far beyond basic element derivations. For instance, the method of solving the equilibrium equations, presents significant difficulties and is greatly influenced by the distribution of the element's nodal degrees of freedom. Other factors which influence the general problem are : the element's degree of accuracy, its adaptability to irregular boundaries and material variations, the input-output requirements, computation time, and the size and configuration of the problems to be solved.

This paper deals with the three-dimensional problem from this general viewpoint. The major parameters in three-dimensional analysis are solution efficiency and accuracy. The paper will discuss the interdependence of these two parameters on the element model, input-output requirements, and the method of solution of the equilibrium equations. It will first be demonstrated

(*) General Electric Co.
(**) Gulf General Atomic..
(***) Gulf Computer Sciences.

that for a given degree of displacement approximation, the selection of the finite element model is not independent of the solution of the equilibrium equations, which is simple verification of the fact that the efficiency of the element is strongly dependent upon the distribution of the element nodal degrees of freedom. Secondly, a finite element model will be described, which provides the desired accuracy but does not overtax the solution process. Thirdly, a well suited method of solution of the equations for the selected model will be presented.

2. Element efficiency

The various types of three-dimensional finite element models have varying influence of the efficiency of the solution of the equilibrium equations. This is due to the distribution of the nodal degrees of freedom of the elements. The efficiency parameter is regarded here as the band width of the system which is the limiting factor in three-dimensional problems of practical interest. In order to illustrate the influence of this element property on the bandwidth we idealize a simple structure by the several element models shown in Table 1, maintaining comparable accuracy in each idealization. The emphasis, here, is on the relationship between the distribution of the nodal degrees of freedom in the assembled element aggregate, provided they are all admissible, and the solution of the equilibrium equations. Therefore, the element shape would not influence the results and remains as a free parameter. Types (a) and (b) shown have identical kinematic properties : they utilize complete third-order polynomial expansion, and they differ only in the type and distribution of the nodal degrees of freedom. But

Table 1
FINITE ELEMENTS COMPARISON

	Element Type	Finite Element Grid in Terms of Main Nodes	Displacement Function	Number of Main Nodes	Number of Subnodes	Number of Degrees of Freedom	Band Width	Number of Tetrahedrons
(a)		3 X 5 X 11	Complete Cubic	165	2656	8463	1641	480
(b)		3 X 5 X 11	Complete Cubic	165	1096	5268	1011	480
(c)		4 X 7 X 16	Complete Quadratic	448	2376	8463	1095	1620
(d)		4 X 7 X 16	Partial Cubic	448	0	5376	673	1620
(e)		7 X 13 X 31	Linear	2821	0	8463	549	12960

element (c) is based on a complete second order polynomial expansion. Interpolation elements fall somewhere between type (c), and (a) or (b). Element (d), which will be derived in this paper, is based on an incomplete third-order polynomial expansion that combines a complete second order polynomial function with six additional cubic terms (*). These element models all have one thing in common, namely, they contain at least linear variations of the strains. This level of accuracy seems to be satisfactory for most problems (*11*). For ease of reference, the four nodes which define the vertices of the tetrahedron are called main nodes, and all others are referred to as subnodes. Consider a structure, the exact shape of which is not important in the present context, but which can be idealized by all five types of elements shown in Table 1. The number of main nodes in each idealization is shown in Table 1. Type (a) element is used as the basis for determining the finite element configurations of the other element types. For example, a cubic variation, such as provided by element (a), can be approximated as three linear functions. Similarly, two cubic segments approximating one curve can be replaced by three parabolas. This is a more meaningful criterion than using the total number of degrees of freedom.

It is interesting to note that the solution time, which for core-resident Gaussian elimination varies as the square of the band width, can vary by a factor of two for types (a) and (b), even though they have identical kinematic properties but differ in the distribution of the nodal degrees of freedom. From the point of view of element efficiency, element (b) is superior to (a). On the other hand if curved elements are used element (b) may present some difficulties because the curvilinear surface coordinates are described in terms of slopes instead of merely nodal coordinate numbers. However, this disadvantage may be overcome by introducing geometrical sub-nodes, two per edge, which are used to describe the geometry only [see Reference (*6*)]. Similar arguments may be applied to elements (c) and (d). Although the latter involves higher-order displacement approximations and consequently more degrees of freedom, it is seen to be more efficient than element (c), even for a similar idealization. In an overall evaluation of all four types, element (b) offers the best kinematic properties while maintaining the bandwidth within manageable limits, and type (d) offers acceptable accuracy, the simplest input requirements, and the highest efficiency. In general, the element with the least number of nodes is the most efficient. In view of the alternating component iterative method (*9, 10*) element (d), or its equivalent, permits a 12-fold increase in the bandwidth capacity over Gaussian elimination, compared with a threefold increase for type (c), (a), or their equivalents.

(*) Similar element has been discussed in Reference 4.

3. Mathematical preliminaries

Let X and Y be two sets of rectangular and skew Cartesian axes, respectively. Then the coordinate transformation

$$y_i = y_i (x_1, x_2, x_3) \qquad i = 1, 2, 3 \tag{1}$$

involves only the direction cosines of the y axes relative to the x axes. If the y_i's are normalized, such that

$$\theta_i = \frac{y_i}{L_{Oi}}, \qquad i = 1, 2, 3 \tag{2}$$

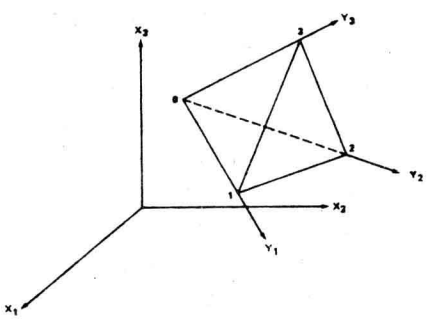

Fig. 1. — Tetrahedral Element, with Twelve Degrees of Freedom per Node (U_i, $\partial U_i/\partial X_j$;i,j, = 1, 2, 3).

where L_{Oi} is the length of edge Oi of the tetrahedron (see Figure 1), then a third order polynomial complete in the θ coordinate system can be written as

$$u(\theta_1, \theta_2, \theta_3) = \langle 1, \theta_1, \theta_2, \theta_3, \theta_1^2, \theta_2^2, \theta_3^2, \theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3, \theta_1^3, \theta_2^3, \theta_3^3,$$

$$\theta_1^2\theta_2, \theta_1\theta_2^2, \theta_2^2\theta_3, \theta_2\theta_3^2, \theta_1^2\theta_3, \theta_1\theta_3^2, \theta_1\theta_2\theta_3 \rangle \begin{Bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_{20} \end{Bmatrix} \tag{3}$$

The θ shape functions can be simply derived in terms of the rectangular Cartezian coordinates x_1, x_2, x_3 as follows. Let γ be defined as

$$\gamma = \langle\ 1,\ x_1,\ x_2,\ x_3\ \rangle \begin{Bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_4 \end{Bmatrix}. \tag{4}$$

Evaluating γ at each of the four vertices of the tetrahedron gives

$$\gamma_0 = D_0\ \alpha, \tag{5}$$

where D_0 is a non-singular 4×4 matrix. Substituting for α from Equation (5) into Equation (4) yields

$$\gamma = \langle\ 1\ x_1\ x_2\ x_3\ \rangle\ [D_0]^{-1}\ \gamma_0. \tag{6}$$

Then the θ's are defined by

$$\langle\ \theta_0\ \theta_1\ \theta_2\ \theta_3\ \rangle = \langle\ 1\ x_1\ x_2\ x_3\ \rangle\ [D_0]^{-1}, \tag{7}$$

where

$$\theta_0 = 1 - \theta_1 - \theta_2 - \theta_3. \tag{8}$$

It is easy to show that upon substituting Equation (7) into Equation (3), the resulting polynomial is also complete in the x_i coordinates. A complete n^{th} order polynomial can also be expressed as a linear combination of all possible product terms of the four θ_i functions such that the order of the products is exactly n.

An alternate form to (3) is then given by

$$u\ (\theta_0, \theta_1, \theta_2, \theta_3) = \langle\ \theta_0^3, \theta_1^3, \theta_2^3, \theta_3^3, \theta_0^2\,\theta_1, \theta_0\,\theta_1^2, \theta_0^2\,\theta_2, \theta_0\,\theta_2^2, \theta_0^2\,\theta_3, \theta_0\,\theta_3^2, \theta_1^2\,\theta_2,$$
$$\theta_1\,\theta_2^2, \theta_1^2\,\theta_3, \theta_1\,\theta_3^2, \theta_2^2\,\theta_3, \theta_2\,\theta_3^2, \theta_0\,\theta_1\,\theta_2, \theta_1\,\theta_2\,\theta_3, \theta_2\,\theta_3\,\theta_0, \theta_3\,\theta_0\,\theta_1\ \rangle \begin{Bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_{20} \end{Bmatrix}. \tag{9}$$

If $u_1\ (\theta_1, \theta_2, \theta_3)$ and $u_2\ (\theta_0, \theta_1, \theta_2, \theta_3)$ are two n^{th} order polynomials of the type shown in Equations (3) and (9), having coefficients α and β respectively, then the following relation exists

$$\alpha = A\,\beta \tag{10}$$

where A is an m × m matrix. If u_1 is complete, then a necessary and sufficient condition for u_2 te be complete is

$$\det |A| \neq 0. \tag{11}$$

It is easy to verify that $\det |A| = \pm 1$ for any n.

4. Element displacement Field

The basis for selecting an admissible polynomial expansion of order n of the displacement field is that it must be continuous through out the solid and must have piecewise continuous derivatives up to and including the n^{th} derivative. Continuity is guaranteed, if the displacement on any face of the tetrahedron is uniquely determined by the nodal degrees of freedom of the element on that face. In addition, the displacement along any edge should be uniquely determined by the nodal degrees of freedom of the element on that edge. These requirements are automatically met, if, for example, complete cubic polynomial representations are selected for element types (a), (b) and (c). However, for incomplete polynomial expansions, such as the one required for element (d), the problem is not so simple. In addition to continuity, which is a global requirement on the displacement field, the displacement functions of each element must satisfy certain fundamental properties dictated by convergence requirements. These can be formulated as a simple rule : first, constant strain states must be fully represented, and secondly, higher-orders terms should be included only if all the lower-order terms are present. A displacement function that satisfies these local requirements and the global continuity requirement is given below for the u_1 displacement component.

$$u_1(x_1, x_2, x_3) = \langle \theta_0^2, \theta_1^2, \theta_2^2, \theta_3^2, \theta_0\theta_1, \theta_0\theta_2, \theta_0\theta_3, \theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3, \theta_0^2\theta_1, \theta_1^2\theta_2, \theta_2^2\theta_3, \theta_3^2\theta_0, \theta_0^2\theta_2, \theta_1^2\theta_3 \rangle \begin{Bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{16} \end{Bmatrix}. \tag{12}$$

The first ten terms of Equation (**12**), by the argument of Section 3, form a complete second-order polynomial. The six cubic terms were selected by experimentation from a rather limited set of alternatives. A property of Equation (**12**) that provides a necessary condition for inter-element compatibility is that it reduces to nine independent terms on each of the four

faces, and to four independent terms on each of the six edges of the tetrahedron. This can be verified easily by substituting the equations of the plane and the line respectively into Equation (**12**). The equation of the plane opposite node i, for example, is simply

$$\theta_i = 0, \tag{13}$$

and the equations of the line opposite nodes i and j are

$$\left.\begin{aligned} \theta_i &= 0 \\ \theta_j &= 0 \end{aligned}\right\}, \quad i \neq j. \tag{14}$$

An obvious disadvantage of Equation (**12**) is that it does not possess nodal symmetry in the contributions of the cubic terms, which introduces some bias in the element's deformation characteristics. However, intuitively and on the basis of limited experience, since the bias is confined to the higher-order terms, it does not have a significant effect, especially in the overall aggregate of elements, where the bias tends to cancel out due to the arbitrary nature of the element orientations.

5. Admissibility of strains as nodal unknowns

Single valued strains at the nodes is not generally admissible in non-homogeneous solids. In most practical problems, however, material inhomogeneities can exist in one of two forms. The first involves a slight variation from region to region due to age, as in concrete structures, for example, or due to temperature dependent properties under nonuniform temperature fields. The second form arises in composite solids where materials of largely different properties are jointed together, such as in welded or reinforced structures. In the former type, the inconsistencies arising from forcing all the strains and rotations to be single-valued at the nodal points are of little significance and can be ignored. In the second class of structures, however, the inconsistencies may not be so easily ignored. In structures where there is distinct separation between the various material constituents, thin layers of elements having average material properties may be inserted at the joints. The inconsistencies are not completely removed, but they somewhat averaged out over small regions. Where this type of approximation is not acceptable, elements of types (a) or (c) should be used.

6. Element stiffness derivations

Instead of discussing the derivation of stiffness matrices and force vectors, which generally follows a standard procedure of matrix algebra we give the quantities of interest directly and discuss the simplifications, made

possible by the particular form of the displacement functions, Equation **(12)**. The element stiffness matrix and force vectors are of the form :

$$K = \int_V A_\varepsilon^T H A_\varepsilon \, dv, \tag{15}$$

$$F_t = \int_v A_\varepsilon^T H E \, dv, \tag{16}$$

$$F_b = \int_v A_u^T B \, dv, \tag{17}$$

$$F_s = \int_\Sigma A_u^T s \, d\Sigma, \tag{18}$$

where ^{T} indicates matrix transpose

- K Stiffness matrix, (48×48)
- H Elasticity matrix, (6×6)
- A_ε Matrix function relating the strains to the nodal unknowns, (6×48)
- A_u Matrix function relating the three displacement components to the nodal unknowns, (3×48)
- E Thermal strain vector function, (6×1)
- B Body force vector function, (3×1)
- S Surface traction vector function, (3×1)
- F_t Nodal force vector due to the tehrmal strains
- F_b Nodal force vector due to the body forces
- F_s Nodal force vector due to the surface tractions
- V Element volume
- Σ Element surface where tractions are specified

Equations **(15)** through **(18)** can be simplified by taking advantage of the properties of the θ functions and further reducing the matrices A_ε and A_u. From the strain-displacement relations

$$\varepsilon = Lu, \tag{19}$$

where L is the well known 6×3 strain-displacement matrix differential operator and ε and u are the strain and displacement vectors, respectively. From Equation **(12)** we have

$$u = \Phi \alpha \tag{20}$$

where Φ is given by

$$\Phi = \begin{bmatrix} \phi & 0 & 0 \\ \theta & \phi & 0 \\ 0 & 0 & \phi \end{bmatrix} \tag{21}$$

and ϕ is the 16 term row vector defined in Equation (**12**). The generalized displacement vector α consists of three sets of α's from Equation (**12**). Upon substituting Equations (**20**) and (**21**) into Equation (**19**), and after some manipulation, we obtain

$$\varepsilon = \psi_x \psi_0 \alpha. \tag{22}$$

Evaluating α in terms of the nodal displacement vector v (displacements and their derivatives) gives

$$\alpha = \Phi_0^{-1} v, \tag{23}$$

where Φ_0 is a block diagonal matrix, consisting of three 16×16 constant matrices obtained by evaluating Φ and its three derivatives at each of the four nodes of the tetrahedron. Equations (**22**) and (**23**) combine to give

$$\varepsilon = \psi_x \psi_0 \Phi_0^{-1} v, \tag{24}$$

from which

$$A_\varepsilon = \psi_x \psi_0 \Phi_0^{-1} \tag{25}$$

Also combining Equations (**20**) and (**23**) gives

$$u = \Phi \Phi_0^{-1} v, \tag{26}$$

which defines A_u as follows :

$$A_u = \Phi \Phi_0^{-1}. \tag{27}$$

The matrices ψ_0 and ϕ_0 are easy to derive and are defined in the appendix. The matrix ψ_x is defined in the following equation :

$$\psi_x = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \tag{28}$$

where

$$\lambda = \langle \theta_0^2, \theta_1^2, \theta_2^2, \theta_3^2, \theta_0\theta_1, \theta_0\theta_2, \theta_0\theta_3, \theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3 \rangle. \quad (29)$$

This is the 10-term complete second order polynomial. Substituting A_ε in Equation (**15**) gives

$$K = [\Phi_0^{-1}]^T \psi_0^T \int_v \psi_x^T H \psi_x \, dv \, \psi_0 \Phi_0^{-1} \quad (30)$$

The matrix function $\psi_x^T H \psi_x$ in Equation (**30**) is a 60×60 symmetric matrix that can be partitioned into 6×6 submatrices, each of which is of size 10 and has the form

$$[\psi_x^T H \psi_x]_{ij} = h_{ij} [\lambda^T \lambda], \; i, j = 1, 2, ..., 6, \quad (31)$$

where the subscripts ij identify the 10×10 submatrices, and h_{ij} are the elements of H. The integration in Equation (**30**) is performed on the matrix $(\lambda^T\lambda)$ only once. This involves volume integrals of the type

$$I_1 = \int_v \theta_i^p \theta_j^q \theta_k^r \, dv = \frac{6 V (p!\, q!\, r!)}{(p + q + r + 3)!}, \quad (32)$$

area integrals over the tetrahedron faces of the type

$$I_2 = \int_A \theta_i^p \theta_j^q \, dA = \frac{2 A (p!\, q!)}{(p + q + 2)!}, \quad (33)$$

and, finally, line integrals of the type

$$I_3 = \int_L \theta_i^p \, dL = \frac{L}{p + 1} \quad (34)$$

In the above expressions

V element volume, $\frac{1}{6}$ det $|D_0|$ [see Equation (**5**)]

p, q, r Integer powers

i, j, k distinct subscripts, 0, 1, 2, 3

A triangular area on which tractions are specified

L length of the element edge on which a line load may be specified

The above integrals may be extended to integrands involving higher products of θ functions by substituting Equation (**8**), (**13**), and (**14**), respectively. This gives

$$\left.\begin{aligned} I_1 &= V/\beta_1 \\ I_2 &= A/\beta_2 \\ I_3 &= L/\beta_3 \end{aligned}\right\} \quad (35)$$

where β_i are functions of p, q, r, and s, and are listed in Table 2.

7. Solution of the equilibrium equations

The method of solution used here is the alterning component iterative method that was presented in detail in References 9 and 10, for a general m-component system (m is the number of nodal degrees of freedom). This method is well suited for the element of four nodes and sixteen degrees of freedom (type d), for which m = 12. The method was adapted to thin-shell problems (*12*) where the nodal unknowns were grouped in four components (m = 4).

This type of grouping, which can be done selectively, proved to be a very convenient feature of the method, where optimum convergence rates can be tailored to the physical problem being solved. As is well known, iterative methods for flexible structures such as thin shells can be hopelessly slow in converging. The rate of convergence of the alternating component method can be greatly influenced, however by simply grouping the unknowns in their order of importance (see Figure 1). Also by permitting m to be an input parameter, component grouping can be varied to accommodate larger band widths, since the bandwidth limitation in this method is B/m, where B is the bandwidth of the total system. It might be of interest to note that m equal to unity implies Gaussian elimination, which, in the present context, is least efficient from the point of view of core storage and solution time. In the other extreme where m can take on the maximum number of nodal degrees of freedom, twelve in the present case, very large bandwidths can be treated. However, in slowly convergent problems, such as in thin shell structures, for example, such luxury in bandwidth is obtained at the expense of the number of iteration cycles. Three-dimensional solids problems solved to date by this method require an average of ten to fifteen cycles. This number may be doubled for thin-shell structures of for ill-conditioned problems. Figure 3 shows solution times for a three-component scheme. In the following we summarize the essential procedure of the alternating-component iterative method adapted to 12 components.

The governing system of equations of a finite element aggregate composed of type (d) elements may be written in the following partitioned matrix form :

$$F_i\,(P) = \sum_{j=1}^{12} K_{ij}\,(P, Q)\,V_j\,(Q) \qquad i = 1, 2, \ldots, 12, \tag{36}$$

where P and Q are lists of field and source points, respectively, $F_i(P)$ and $V_i(P)$ are load and displacement subvectors, respectively, and $K_{ij}(P, Q)$ are the stiffness coefficient matrices of order N where N is the number of nodal

points. The block over-relaxation iterative form of Equation (**36**) is given by

$$V_i(P)^{(s+1)} = V_i(P)^{(s)} + \omega K_{ii}^{-1}(P, Q)\left[F_i(P) - \sum_{j=1}^{i-1} K_{ij}(P, Q)\, V_j(Q)^{(s+1)} - \right.$$

$$\left. - \sum_{j=i}^{12} K_{ij}(P, Q)\, V_j(Q)^{(s)}\right] \qquad i = 1, 2, \ldots, 12, \tag{37}$$

where $1 \leqslant \omega < 2$ is the over-relaxation factor and s is the iteration cycle. In order to make further use of Equation (**37**), we take advantage of the bandedness of the system and assume that the list of mesh points P can be partitioned into M point groups P_1, P_2 ..., P_M, where coupling of one group extends to the two adjacent groups only. Equation (**36**) can be partitioned further as follows:

$$F_i(P_\alpha) = \sum_{j=1}^{12} \sum_{\beta=1}^{M} K_{ij}(P_\alpha, Q_\beta)\, V_j(Q_\beta) \qquad \begin{matrix} \alpha = 1, 2, \ldots, M \\ i = 1, 2, \ldots, 12. \end{matrix} \tag{38}$$

TABLE 2. — Volume, Surface and Line Integrals

p	q	r	s	β_1	β_2	β_3
0	0	0	0	1	1	1
1	0	0	0	4	3	2
2	0	0	0	10	6	3
1	1	0	0	20	12	6
3	0	0	0	20	10	4
2	1	0	0	60	30	12
1	1	1	0	120	60	—
4	0	0	0	35	15	5
3	1	0	0	140	60	20
2	2	0	0	210	90	30
2	1	1	1	420	—	—
1	1	1	1	840	—	—
2	1	1	0	—	180	—

In this form, the component matrices $K_{ij}(P, Q)$ in Equation (**38**) are block-tridiagonal, i.e., $K_{ij}(P_\alpha, Q_\beta)$ are non-zero for $(\alpha - 1) \leqslant \beta \leqslant (\alpha + 1)$. Further, in order to avoid confusing notation, Equation (**37**) may be written

$$V_i(P)^{(S+1)} = V_i(P)^{(S)} + \omega \Delta V_i(P)^{(S+1)} \tag{39}$$

in which

$$\Delta V_i(P)^{(S+1)} = K_{ii}^{-1}(P, Q)\, R_i(Q)^{(S+1)}, \tag{40}$$

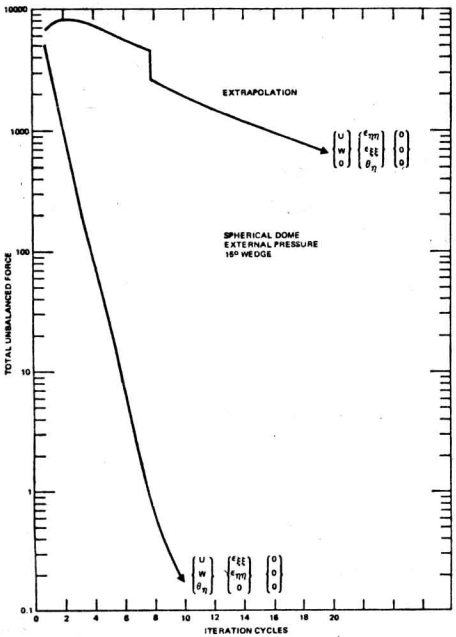

Fig. 2. — Influence of Component Grouping on the Rate of Convergence.

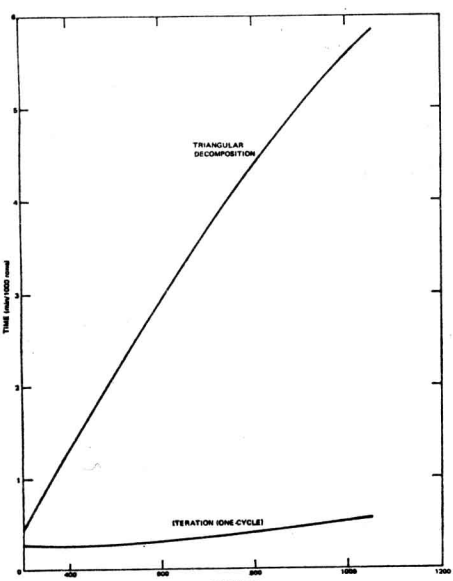

Fig. 3. — Equations Solution Time for a Three-component Scheme on UNIVAC-1108 Single Precision.