HIGHER COMBINATORICS

edited by

MARTIN AIGNER

NATO ADVANCED STUDY INSTITUTES SERIES

Series C: Mathematical and Physical Sciences

Higher Combinatorics

Proceedings of the NATO Advanced Study Institute held in Berlin (West Germany), September 1-10, 1976

edited by

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Königin-Luise-Strasse 24/26, D-1000 Berlin 33, B.R.D.

D. Reidel Publishing Company

Dordrecht-Holland/Boston-U.S.A.

Published in cooperation with NATO Scientific Affairs Division

ISBN 90-277-0795-2

Published by D. Reidel Publishing Company
P.O. Box 17, Dordrecht, Holland

Sold and distributed in the U.S.A., Canada, and Mexico
by D. Reidel Publishing Company, Inc.
Lincoln Building, 160 Old Derby Street, Hingham, Mass. 02043, U.S.A.

Printed in The Netherlands

PREFACE

It is general consensus that Combinatorics has developed into a full-fledged mathematical discipline whose beginnings as a charming pastime have long since been left behind and whose great significance for other branches of both pure and applied mathematics is only beginning to be realized. The last ten years have witnessed a tremendous outburst of activity both in relatively new fields such as Coding Theory and the Theory of Matroids as well as in more time honored endeavors such as Generating Functions and the Inversion Calculus. Although the number of text books on these subjects is slowly increasing, there is also a great need for up-to-date surveys of the main lines of research designed to aid the beginner and serve as a reference for the expert.

It was the aim of the Advanced Study Institute "Higher Combinatorics" in Berlin, 1976, to help fulfill this need. There were five sections: I. Counting Theory, II. Combinatorial Set Theory and Order Theory, III. Matroids, IV. Designs and V. Groups and Coding Theory, with three principal lecturers in each section. Expanded versions of most lectures form the contents of this book. The Institute was designed to offer, especially to young researchers, a comprehensive picture of the most interesting developments currently under way. It is hoped that these proceedings will serve the same purpose for a wider audience.

Sincere thanks are due to the Freie Universität Berlin and, in particular, to the North Atlantic Treaty Organization for their generous support which made this Institute possible. It is our hope that after Nijenrode 1974 and Berlin 1976 there will be another Advanced Study Institute on Combinatorics in the near future.

Berlin, October 1976 Martin Aigner

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TABLE OF CONTENTS

PART I

COUNTING THEORY

ENUMERATION OF PARTITIONS: THE ROLE OF EULERIAN SERIES AND q-ORTHOGONAL POLYNOMIALS

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ABSTRACT. The theory of partitions has long been associated with so called basic hypergeometric functions or Eulerian series. We begin with discussion of some of the lesser known identities of L.J. Rogers which have interesting interpretations in the theory of partitions. Illustrations are given for the numerous ways partition studies lead to Eulerian series. The main portion of our work is primarily an introduction to recent work on orthogonal polynomials defined by basic hypergeometric series and to the applications that can be made of these results to the theory of partitions. Perhaps it is most interesting to note that we deduce the Rogers-Ramanujan identities from our solution to the connection coefficient problem for the little q-Jacobi polynomials.

1. SOME THEOREMS OF L.J. ROGERS

L.J. Rogers [20] became famous for the following two identities that were proved by him in 1894.

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(1) Partially supported by the United States Army under Contract No. DAAG29-75-C-0024 and National Science Foundation Grant MSP74-07282.
(2) Partially supported by the United States Army under Contract No. DAAG29-75-C-0024 and National Science Foundation Grant MCS75-06687-3.

M. Aigner (ed.), Higher Combinatorics, 3-26.

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \tag{1.1}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{1.2}$$

The story of the neglect of these identities and their subsequent rediscovery by S. Ramanujan has been told many times [17; pp.90-91], [8; Ch.7], so we shall not dwell upon it here. Our current object is to examine several other results of Rogers that are still little known. These identities are nearly as striking as the "Rogers-Ramanujan identities" (i.e. (1.1) and (1.2)) and their partition-theoretic counterparts are even more surprising. These results and the general discussion of such problems in Section 2 lead us to the conclusion that an extensive investigation of the analytic theory of Eulerian series is long overdue. In Section 3, we make some contributions to such a study by illustrating how orthogonal polynomials defined by basic hypergeometric functions are useful in the study of partitions; the results in this Section are merely an introduction to an extensive study of such orthogonal polynomials in [9].

1.1 The analytic form of Rogers's identities

Rogers (in [20] and [21]) gave numerous series-product identities. To avoid dilution of the interest of these results by exhibiting too many, we choose six which well illustrate these amazing discoveries of Rogers:

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^{2n})}$$

$$=\prod_{n=0}^{\infty}\frac{1}{(1-q^{2n+1})(1-q^{20n+4})(1-q^{20n+16})}, \tag{1.1.1}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+2n}}{(1-q)(1-q^2)\dots(1-q^{2n+1})}$$

$$=\prod_{n=0}^{\infty}\frac{1}{(1-q^{2n+1})(1-q^{20n+8})(1-q^{20n+12})}, \tag{1.1.2}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^{2n})}$$

$$=\prod_{\substack{n=1\\ n\not\equiv\pm1,\pm8,\pm9,10\ (\mathrm{mod}\ 20)}}^{\infty}(1-q^n)^{-1}, \qquad (1.1.3)$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}$$

$$=\prod_{\substack{n=1\\ n\not\equiv\pm3,\pm4,\pm7,10\ (\mathrm{mod}\ 20)}}^{\infty}(1-q^n)^{-1}; \qquad (1.1.4)$$

$$\Big\{\prod_{m=1}^{\infty}(1+q^{2m})\Big\}\Big\{1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}\Big\}$$

$$=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}; \qquad (1.1.5)$$

$$\Big\{\prod_{m=1}^{\infty}(1+q^{2m})\Big\}\Big\{1+\sum_{n=1}^{\infty}\frac{q^{n^2+2n}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}\Big\}$$

$$=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \qquad (1.1.6)$$

For reference we note that (1.1) and (1.2) first appeared in [20; p.328, eq.(1) and p.329, eq.(2)]. Equation (1.1.1) is the second equation on page 331 of Roger's memoir [20], while equation (1.1.2) is equivalent to equation (7) on page 331 of [20]. Equation (1.1.3) is equivalent to the equation appearing just after equation (12) on page 332 of [20], and equation (1.1.4) is equivalent to equation (6) on page 331 of [20]. Equation (1.1.5) is the last equation on page 330 of [20], and equation (1.1.6) is the identity preceding equation (7) on page 331 of [20].

1.2 The partition-theoretic implications of Rogers's identities.

Section 2 will discuss the actual techniques used to shift from combinatorial to analytic identities. Here we confine ourselves to an examination of the quite unexpected partition theorems that have arisen from (1.1.1)-(1.1.6).

We begin by recalling

Theorem 1. (Euler's theorem). The number of partitions $b_1+b_2+\ldots+b_r$ of n where $b_1 \ge b_2 \ge b_3 \ge \ldots$ and each b_i is odd equals the number of partitions $c_1+c_2+\ldots+c_s$ of n where $c_1 > c_2 > c_3 > \ldots$.

A standard exercise in elementary partition theory (cf. [3; Ch.13],[18; Ch.19]) shows that Euler's theorem is easily derived from the trivial infinite product identity

$$\prod_{n=1}^{\infty} (1+q^n) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} .$$

B. Gordon [16] observed that (1.1.1) may be interpreted as follows:

Theorem 2. (Gordon's theorem). The number of partitions $b_1+b_2+\ldots+b_r$ of n where $b_1 \ge b_2 \ge b_3 \ge \ldots$ and each b_i is odd or $\equiv \pm 4 \pmod{20}$ equals the number of partitions $c_1+c_2+\ldots+c_3$ of n where $c_1 > c_2 \ge c_3 > c_4 \ge c_5 > \ldots$.

Thus Theorem 2 seems quite closely related to Theorem 1; however its proof lies much deeper. There is a similar interpretation of (1.1.2) mentioned by Gordon [16] and explicitly given by W. Connor [13]:

Theorem 3. The number of partitions $b_1+b_2+\ldots+b_r$ of n where $b_1 \ge b_2 \ge b_3 \ge \ldots$ and each b_i is odd or $\equiv \pm 8 \pmod{20}$ equals the number of partitions $c_1+c_2+\ldots+c_{2s+1}$ of n into an odd number of parts where $c_1 \ge c_2 \ge c_3 > c_4 \ge c_5 > c_6 \ge c_7 > \ldots$ (i.e. $c_i \ge c_{i+1}$ with strict inequality if i is odd and ≥ 3).

W. Connor [13] has also interpreted (1.1.3) and (1.1.4):

Theorem 4. The number of partitions $b_1+b_2+\ldots+b_r$ of n where $b_1 \ge b_2 \ge b_3 \ge \ldots$ and each b_i is $\not\equiv \pm 1, \pm 8, \pm 9, 10 \pmod{20}$ equals the number of partitions $c_1+c_2+\ldots+c_s$ of n into an even number of parts where $c_1 \ge c_2 > c_3 \ge c_4 > c_5 \ge \ldots$.

Theorem 5. The number of partitions $b_1+b_2+\ldots+b_r$ of n where $b_1 \ge b_2 \ge b_3 \ge \ldots$ and each $b_i \not\equiv \pm 3, \pm 4, \pm 7, 10 \pmod{20}$ equals the number of partitions $c_1+c_2+\ldots+c_s$ of n where $c_1 \ge c_2 > c_3 \ge c_4 > c_5 \ge c_6 > \ldots$.