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Edited by A. Dold and B. Eckmann

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Elliptic Boundary Value Problems on Corner Domains

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Smoothness and Asymptotics of Solutions

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FOREWORD

Many physical phenomena are described by elliptic boundary value problems : let us quote vibrating membranes, elasticity, electrostatics, hydrodynamics for instance. Natural domains are often non-smooth ones or they may be "small perturbations" of such non-regular domains. That is why many people are interested in domains with singularities on their boundaries.

In this book, we deal with a great variety of domains : we consider conical singularities of course, but also edges, polyhedral corners, combined with various types of cracks, holes or slits.

In order to give precise mathematical results, we need to choose a functional framework. So we decided, therefore to choose ordinary hilbertian Sobolev spaces with real exponents (also called Sobolev-Slobodeckii spaces). Other choices are possible, but we prefered this one for several reasons that we explain in the introduction .

We develop a general theory : first, we characterize different fundamental properties of induced operators, in particular regularity, Fredholm and semi-Fredholm properties, and then we give asymptotics of solutions in the neighborhood of singular points of the boundary.

Our results can be applied to specific problems : in such cases, it is often possible to get the characteristic conditions we give more precise. As an example, we do this for the Dirichlet problem associated to the Laplace equation. In another paper, we apply them to the Stokes system.

Moreover, the type of statements we get can be adapted to other problems than those we consider here : for instance to non-homogeneous boundary data, to lifting of traces, and also to the study of such problems in other classes of hilbertian weighted Sobolev spaces.

So our results can be used in direct or indirect ways. More introductory details may be found in the preface and in the first section.

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Nantes,
March 5th, 1988.

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INTRODUCTION

In this book, we intend to study properties of elliptic boundary value problems on n-dimensional domains with corners. The singular points of the domains may be of various types : conical points, edges, cuts, slits, polyhedral corners, and so on... We will be mainly concerned with 2m-order boundary value problems that arise from variational formulation : they stem from integro-differential coercive forms with smooth coefficients on a subspace V of the Sobolev space H^m. Such a problem induces operators acting on Sobolev spaces H^s, with s denoting any real number greater than m. (Here, we only consider Hilbert spaces). We state necessary and sufficient general conditions characterizing different properties of these operators : closed range, index or regularity properties. In some situations, we are able to describe the asymptotics of the solutions near the singular points of the boundary. Moreover, in some particular cases, we explicit the general conditions quoted above.

Many works are devoted to boundary value problems on non-smooth domains : see the bibliography and also the survey [KO-OL] and the book [GR 6] whose references are more comprehensive. Thus the theory of such problems is interesting for itself ! Moreover, the knowledge of the structure of solutions is useful in numerical analysis by the finite element method : estimates in Sobolev spaces are essential to determine the convergence rate when the parameter h of the triangulation tends to zero (see [CI] for the general theory and, for example, [SC-WA], [RA], [LA], [BDLN], [AU] for more particular problems on polygons).

Our work develops many results, which are complementary to already known results. It also constitutes a synthesis of some of these results.

Before going on, let us situate our work in comparison with already known results.

Roughly speaking, results about corner problems can be divided into two types : on the one hand, the precise study of a particular type of operators on special domains (e. g. the Laplace operator or the Stokes system on a polygonal domain) ; on the other hand, the general theory about any operators, on various domains. P. Grisvard [GR1-6] gives a lot of contributions about precise results, as many other authors. V.A. Kondrat'ev [KO 1-3] and V.G. Maz'ja & B.A. Plamenevskii [MA-PL 1-4] are the main contributors for the general theory. Another interesting, but more abstract, point of view is the construction of parametrices in various algebras of pseudodifferential operators : this is performed by R. Melrose and G. Mendoza [ME-ME], and by S. Rempel and B.W. Schulze (e.g. [RM-SU], [SU]), for totally characteristic or degenererate operators on conical manifolds without boundary. This is not exactly in the scope of our work.

Here is a short description of the works [KO 1-3] and [MA-PL1-4]. The basic paper about boundary value problems on domains with conical points is Kondrat'ev's in [KO 1]. The main contributions about edges and polyhedral singularities are those of Maz'ja and Plamenevskii in [MA-PL 3-4].

In order to classify these results, we specify the functional spaces used for them. As was already said, we limit ourselves to Hilbert spaces. Before going on, however, we emphasize the powerful method of [MA-PL1,2] for L_p Sobolev spaces and Hölder classes.

The functional spaces divide into three types :

- type (S) : ordinary Sobolev spaces H^k :

$$u\in H^k \iff D^\alpha u\in L^2,\ \forall\alpha\in\mathbf{N}^n,\ |\alpha|\leqslant k\ ;$$

- type (K) : totally characteristic weighted spaces H^k_γ

$$u\in H^k_\gamma \iff r^{\gamma+|\alpha|-k}D^\alpha u\in L^2,\ \forall\alpha\in\mathbf{N}^n,\ |\alpha|\leqslant k$$

where r is the distance from the singular points of the boundary ;

- type (W) : weighted spaces W^k_γ

$$u\in W^k_\gamma \iff r^\gamma D^\alpha u\in L^2,\ \forall\alpha\in\mathbf{N}^n,\ |\alpha|\leqslant k.$$

γ is in each case a real number, and k is a non-negative integer.

We summarize some information in a table (see the next page). Let us also quote the work of P. Bolley and J. Camus [BO-CA 3] in (W)-spaces for boundary-degenerate operators (now, r is the distance to the boundary). G. Eskin [ES 1, 2] also uses (W)-spaces for elliptic boundary value problems for second order operators in polygons or in domains with edges. His $H_{s,N}$ spaces on a two-dimensional domain with a corner in 0 are defined as follows : for $s\in\mathbf{R}$ and $N\in\mathbf{N}$:

$$u \in H_{s,N} \iff \forall k, \ell \in \mathbf{N} \;\; k+\ell \leqslant N \;\; x_1^k x_2^\ell u \in H^{s+k+\ell}.$$

Thanks to Hardy's inequality, it is possible to show that, if $s \in \mathbf{N}$, $H_{s,N}$ coincides with W_N^{s+N}.

Here is our table :

		CONES	Edges and corners
[KO]	spaces results references	(K) and (S) spaces index, singularities [KO 1]	(K) and (W) spaces regularity, singularities (*) [KO 2,3]
[MA - PL]	spaces results references	(K) and (W) spaces index, singularities [MA - PL 1,2]	(K) spaces index [MA - PL 3,4]

(*) for second order operators with real coefficients.

Let us point out that type (S) is a particular case of type (W). Nevertheless, $\gamma=0$ is often a difficult limit case, and is not always a straightforward consequence of the study of (W)-spaces.

Kondrat'ev uses (K)-spaces as preliminary to (S)-spaces. Pham The Lai in [PH] showed that the Mellin transform helps to link (K)-spaces to (S)-spaces and induces a more direct method to exhibit singular solutions. The Mellin transform for corner problems was already used in [BA-SJ] and was used anew in [DA 2] and [BCD] for the analytic or Gevrey regularity in conical domains. In the present work, we succeed in treating the limit case by the Mellin transform.

There were some gaps in the general theory :

❶ the Sobolev exponents are only non-negative integers ;

❷ as soon as edges are present, the functional spaces are only (K)-spaces ;

❸ the closed range property is investigated only for the Laplace operator.

We hope we have completed the general theory with respect to these three axes. We consider that the advantages are the following ones :

① Real (and possibly negative) exponents allow the construction of a continuous chain of spaces linking the space V (where we have the existence of variational solutions) and the

space H^{2m} (where Neumann boundary conditions make sense). In connection with that, we were led to complete the theory about smooth domains too. Thus we get new regularity results for non-Dirichlet value problem on domains in $\mathbf{R}^n$ with $n \geqslant 3$.
On the other hand, the even dimensions are no longer special as they were in [KO 1].

② Non-weighted Sobolev spaces seem more natural :

• right hand sides need not have a particular behavior near corner points (for instance, look at G. Fichera's comments about physical problems in the introduction of [FI]) ;

• the space V need not be a (K)-space : e.g. for Neumann problem when $m \geqslant n/2$.

On the other hand, H^s-spaces are useful in the following situations :

• estimates for convergence rate in finite element method : H^s-estimates in [BDLN], [AU], L^∞ estimates in [SC-WA] ; let us emphasize these regularity results in H^s-spaces for any suitable real number s, allow to take advantage of the full power of Aubin-Nitsche lemma (see §9 in [BDLN]).

• (non-linear) Navier-Stokes system :

•• regularity of solutions : [KE-OS] in polygons, [DA 3] in 2- or 3-dimensional domains ;

•• successive approximation scheme in hydrodynamics [PO-TO].

③ Coercivity properties may arise from closed range estimates.

The characterization of regularity or index properties is usually related to the location of eigenvalues λ of an analytic operator family (cf [KO] and [MA-PL]). We introduce a new condition (C^*) consisting of "injectivity modulo polynomials" at λ, which also depends on the same spectral parameter λ. When λ is not a positive integer, the usual condition (C) "λ is not an eigenvalue", and the new one are equivalent to each other. On the contrary, if λ is a natural integer, the new condition (C^*) is the right one while condition (C) is unnecessary when the domain has a cut, or a slit, or when the boundary problem is the Neumann problem. Besides, condition (C) is generally not sufficient for conical domains in $\mathbf{R}^n$, $n \geqslant 3$. Condition (C) actually suits well to (K)-spaces, whereas condition (C^*) is better for (S)-spaces (and for (W)-spaces that we intend to investigate in another work, for they are useful for collocation methods, see [BN-MA]).

This book is divided into four parts.

The *first part* consists of Chapter one, split into two paragraphs. In §1, we give some precise statements of ours, and we compare them with already known results from [KO1], [MA-PL 3,4] , [HA-SM], and others. In §2, we introduce the classes of corner domains we are interested in. As in [MA-PL4] this is based upon a recurrence on the dimension.

The *second part* is devoted to the Dirichlet problem. In Chapter 2, we introduce some fundamental definitions about frozen operators (§3), we compare the injectivity modulo polynomials and the spectral condition (§4) and we state our general results about regularity, index and closed range (§5,6,7). Proofs are defered to Chapter 3 (§8-12). In Chapter 4, we study the two-dimensional domains (§13), and especially second-order operators with complex coefficients (§14) and also the behavior of a fourth-order operator near a cut or a hole (§15). In Chapter 5, we closely examine singularities along an edge (§16) and in the neighborhood of a three-dimensional polyhedral vertex (§17), for homogeneous operators with constant coefficients. In Chapter 6, we apply our general results to the Laplace operator, and due to the special form of this operator, we are able to state accurate results (§18 for Fredholm properties and §19 for semi-Fredholm ones).

The *third part* is devoted to variational boundary value problems. We deal with Dirichlet problem apart from general problems because it is less difficult and we are able to consider a wider class of domains than we do for general problems. It also allows us to handle separately with the different difficulties in the proofs. Variational boundary problems are considered on polyhedral domains, which allows us to write a Green's formula in order to choose stable or transversal boundary conditions. In Chapter 7, we recall and improve results about smooth domains (statements in §20, proofs in §21) and, in Chapter 8, we deal with boundary problems on polyhedral domains : we formulate the considered problems in §22, we state our results and we give hints about possible extensions in §23, and we develop proofs in §24.

The *fourth part* consists of four separate appendices of which the longest one is appendix A devoted to supplementary background on ordinary or weighted Sobolev spaces, Mellin transform and limit case of Sobolev's embedding theorem. In appendix B, we prove a useful formula linking index and singular functions. In appendix C, we state a result which will be used when we study closed range properties. Finally, in appendix D, we study the ideals of polynomials which are equal to zero on the boundary of an open cone and we prove that such ideals are principal. This allows us to compare conditions (C) with conditions (C^*) when the parameter λ is an integer.

Some of the results of this book were announced in the three notes [DA 6].

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§1 Some results, compared with well-known statements

We recall that the survey [KO-OL] and the book [GR6] contain more comprehensive bibliographies.

For the sake of simplicity, in this paragraph, we limit ourselves to Dirichlet problem. In the third part, we will give other references and explain the additional difficulties of general boundary value problems.

Let Ω be a bounded domain in $\mathbf{R}^n$, and let L be a strongly elliptic 2m-order differential operator on $\overline{\Omega}$, with coefficients in $C^\infty(\overline{\Omega})$. The Dirichlet problem for L induces the following operators.

$$L^{(s)} : \mathring{H}^m \cap H^{s+m}(\Omega) \longrightarrow H^{s-m}(\Omega)$$
$$u \longrightarrow Lu$$

where s is a positive real number and $\mathring{H}^m$ denotes the closure of $\mathscr{D}(\Omega)$ in the usual Sobolev space $H^m(\Omega)$.

1.A Smooth domains and corner domains

When Ω has a smooth boundary the situation is well-known : J.L. Lions and E. Magenes in [LI-MA] prove that if s does not belong to the set {1/2,3/2,..., m-1/2}, then $L^{(s)}$ is a Fredholm operator : i.e. its kernel and its cokernel are finite dimensional ; moreover, we have the following regularity result :

(1.1) if $u \in \mathring{H}^m(\Omega)$ and $Lu \in H^{s-m}(\Omega)$, then $u \in H^{s+m}(\Omega)$.

For all that, it is sufficient that L is proper elliptic. If moreover, L is strongly elliptic, the index of $L^{(s)}$ is zero i.e. :

$$\dim \operatorname{Ker} L^{(s)} = \operatorname{codim} \operatorname{Rg} L^{(s)},$$

where, as usual, Ker A denotes the kernel of the operator A and Rg A denotes its range.

On the contrary, when the domain has corner singularities on its boundary, the regularity property (1.1) no longer holds for all exponents s.

Our main purpose consists in characterizing the s for which (1.1) still holds, the s for which the range of $L^{(s)}$ is a closed subspace of $H^{s-m}(\Omega)$, and the s for which the codimension of Rg $L^{(s)}$ is finite (as, in our cases, Ker $L^{(s)}$ will be finite dimensional, $L^{(s)}$ will