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Serge Lang   William Cherry

Topics in Nevanlinna Theory

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PART ONE

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by Serge Lang

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PART ONE

LECTURES ON NEVANLINNA THEORY

by Serge Lang

CHAPTER I

NEVANLINNA THEORY IN ONE VARIABLE

CHAPTER II

EQUIDIMENSIONAL HIGHER DIMENSIONAL THEORY

PART TWO

NEVANLINNA THEORY OF COVERINGS

by William Cherry

CHAPTER III

NEVANLINNA THEORY FOR MEROMORPHIC FUNCTIONS ON COVERINGS OF C

CHAPTER IV

EQUIDIMENSIONAL NEVANLINNA THEORY ON COVERINGS OF C^n

INTRODUCTION

These are notes of lectures on Nevanlinna theory, in the classical case of meromorphic functions (Chapter I) and the generalization by Carlson-Griffiths to equidimensional holomorphic maps $f\colon \mathbf{C}^n \to X$ where X is a compact complex manifold (Chapter II). Until recently, no special attention was paid to the significance of the error term in Nevanlinna's main inequality, see for instance Shabat's book [**Sh**]. In [**La 8**] I pointed to the existence of a structure to this error term and conjectured what could be essentially the best possible form of this error term in general. I also emphasized the importance of determining the best possible error term for each of the classical functions. I shall give a more detailed discussion of these problems in the introduction to Chapter I. In this way, new areas are opened in complex analysis and complex differential geometry. I shall also describe the way I was inspired by Vojta's analogy between Nevanlinna Theory and the theory of heights in number theory.

P.M. Wong used a method of Ahlfors to prove my conjecture in dimension 1 [**Wo**]. In higher dimension, there was still a discrepancy between his result and the one in [**La 8**], neither of which contained the other. By an analysis of Wong's proof, I was able to make a certain technical improvement at one point which leads to the desired result, conjecturally best possible in general. Using Wong's approach, I was also able to give the same type of structure to the error term in Nevanlinna's theorem on the logarithmic derivative. As a result, it seemed to me useful to give a leisurely exposition which might lead people with no background in Nevanlinna theory to some of the basic problems which now remain about the error term. The existence of these problems and the possibly rapid evolution of the subject in light of the new viewpoints made me wary of writing a book, but I hope these lecture notes will be helpful in the meantime, and will help speed up the development of the subject. They might very well be used as a continuation for a graduate course in complex analysis, also leading into complex differential geometry. Sections 1 and 2 of Chapter I provide a natural bridge, and Chapter I is especially well suited to be used in conjunction with a course in complex analysis, to give applications for the Poisson and Jensen formulas which are usually proved at the end of such a course.

I have not treated Cartan's theorem, giving a second main theorem for holomorphic maps $f: \mathbf{C} \to \mathbf{P}^n$, because I gave Cartan's proof in [**La 7**], in a self contained way, and it is still the shortest and clearest. However, at the end of Chapter II, I give one application of the techniques to one case of a map $f: \mathbf{C} \to Y$ into a possibly non compact complex manifold as an illustration of the techniques in this case. I also have not given the theory of derived curves, which introduced complications of multilinear algebra in Cartan and Ahlfors [**Ah**], and prevented seeing more clearly certain phenomena having to do with the error term, which form our main concern here. I also want to draw attention to Vojta's result for maps $f: \mathbf{C} \to \mathbf{P}^n$ into projective space. In Cartan, Ahlfors, and Schmidt's version (the number theoretic case), it is assumed that the image of f does not lie in any hyperplane. Vojta was able to weaken this assumption to the image of f not lying in a finite union of hyperplanes, which he determines explicitly as generalized diagonals [**Vo 2**]. This result gives substantial new insight into the structure of the "exceptional set" in the linear case. Ultimately, this and other advances will also have to be included into a more complete book account of the theory, as it is now developing.

Serge Lang

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S.L.

TERMINOLOGY AND BASIC NOTATION

By **increasing** we shall mean weakly increasing throughout, so an increasing function is allowed to be constant. **Positive** will mean strictly positive.

The open **disc** of radius R centered at the origin is denoted $\mathbf{D}(r)$. The **circle** of radius r centered at the origin is $\mathbf{S}(r)$. The closed disc is $\overline{\mathbf{D}}(r)$.

In $\mathbf{C}^n$, the **ball** and **sphere** of radius r are denoted

$$\mathbf{B}(r),\ \overline{\mathbf{B}}(r),\ \text{and}\ \mathbf{S}(r)$$

respectively.

Let F_1, F_2 be two positive functions defined for all real numbers $\geq r_0$. We write

$$F_1 \ll F_2$$

to mean that $F_1 = O(F_2)$. We shall write

$$F_1 \gg\ll F_2$$

to mean that $F_1 \ll F_2$ and $F_2 \ll F_1$.

CHAPTER I

NEVANLINNA THEORY IN ONE VARIABLE

In the first part of this chapter we essentially follow Nevanlinna, as in his book [**Ne**]. The main difference lies in the fact that we are careful about the error term in Nevanlinna's main theorem. That this error term has an interesting structure was first brought up in [**La 8**], in analogy with a similar conjecture in number theory. Although Osgood [**Os**] did notice a similarity between the 2 in the Nevanlinna defect and the 2 in Roth's theorem, Vojta gave a much deeper analysis of the situation, and compared the theory of heights in number theory or algebraic geometry with the Second Main Theorem of Nevanlinna theory.

In [**La 2**] and [**La 3**] I defined a **type** for a number α to be an increasing function ψ such that

$$-\log\left|\alpha - \frac{p}{q}\right| - 2\log q \leq \log \psi(q)$$

for all but a finite number of fractions p/q in lowest form, $q > 0$. The **height** $h(p/q)$ is defined as

$$\log \max(|p|, |q|).$$

If p/q is close to α then $\log q$ has the same order of magnitude as the height, so $\log q$ is essentially the height in the above inequality. A theorem of Khintchine states that almost all numbers have type $\leq \psi$ if

$$\sum \frac{1}{q\psi(q)} < \infty.$$

The idea is that algebraic numbers behave like almost all numbers, although it is not clear a priori if Khintchine's principle will apply without any further restriction on the function ψ. Roth's theorem can be formulated as saying that an algebraic number has type $\leq q^{\varepsilon}$ for every $\varepsilon > 0$, and in the sixties I conjectured that this could be improved to having type $\leq (\log q)^{1+\varepsilon}$ in line with Khintchine's principle. Cf. [**La 1**], [**La 3**], [**La 4**] especially.[1] Thus for instance we would have the improvement of Roth's inequality

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C(\alpha, \varepsilon)}{q^2 (\log q)^{1+\varepsilon}}$$

which could also be written

$$-\log\left|\alpha - \frac{p}{q}\right| - 2\log q \leq (1+\varepsilon)\log\log q$$

for all but a finite number of fractions p/q. However, except for quadratic numbers that have bounded type, there is no known example of an algebraic number about which one knows that it is or is not of type $(\log q)^k$ for some number $k > 1$. It becomes a problem to determine the type for each algebraic number and for the classical numbers. For instance, it follows from Adams' work [**Ad 1**], [**Ad 2**] that e has type

$$\psi(q) = \frac{C \log q}{\log\log q}$$

with a suitable constant C, which is much better than the "probabilistic" type $(\log q)^{1+\varepsilon}$.

In light of Vojta's analysis, it occurred to me to transpose my conjecture about the "error term" in Roth's theorem to the context of Nevanlinna theory, in one and higher dimension. Transposing to the analytic context, it becomes a problem to determine the "type" of the

[1] Unknown to me until much later, similar conjectures were made by Bryuno [**Br**] and Richtmyer, Devaney and Metropolis [**RDM**], see [**L-T 1**] and [**L-T 2**].

classical meromorphic functions, i.e. the best possible error term in the second main theorem which describes the value distribution of the function. It is classical, and easy, for example, that e^z has bounded type, i.e. that the error term in the Second Main Theorem is $O(1)$. Two problems arise here:

- To determine for "almost all" functions (in a suitable sense) whether the type follows the pattern of Khintchine's convergence principle.
- To determine the specific type for each concrete classical function, using the specific special properties of each such function: $\wp, \theta, \Gamma, \zeta, J$, etc.

I am much indebted to Ye for an appendix exhibiting functions of type corresponding to a factor of $1 - \varepsilon$ in number theory. Until he gave these examples, I did not even know a function which did not have bounded type.

In [**La 8**], using the singular volume form of Carlson-Griffiths or a variation of it, I was not able to prove my conjecture exactly, with the correct factor of $1 + \varepsilon$ (I got only $3/2$ instead of 1).

P.M. Wong brought back to life a method which occurs in Ahlfors' original 1941 paper, and by this method he established my conjecture with the $1 + \varepsilon$. I pointed out to him that his method would also prove the desired result with an arbitrary type function satisfying only the convergence of the integral similar to the Khintchine principle. The role of the convergence principle becomes very clear in that method, which is given in the second part of this chapter. The method had also been tried improperly by Chern in the early sixties, and we shall have more to say on the technical aspects when we come to the actual theorems in §4. Ahlfors' method was obscured for a long time by other technical aspects of his paper, and I think Wong made a substantial contribution by showing how it could be applied successfully. I should also note that H. Wu also proved the conjecture with $1+\varepsilon$ (unpublished) by the "averaging method" of Ahlfors. But the method used by Wong lent itself better to give the generalized version with the function ψ.

Developing fully the two problems mentioned above would create a whole new area of complex analysis, digging into properties of meromorphic functions in general, and of the classical functions in particular, which up to now have been disregarded.

I, §1. THE POISSON-JENSEN FORMULA AND THE NEVANLINNA FUNCTIONS

By a meromorphic function we mean a meromorphic function on the whole plane, so its zeros and poles form a discrete set. A meromorphic function on a closed set (e.g. the closed disc $\overline{\mathbf{D}}(R)$) is by definition meromorphic on some open neighborhood of the set.

Theorem 1.1. (Poisson formula) *Let f be holomorphic on the closed disc $\overline{\mathbf{D}}(R)$. Let z be inside the disc, and write $z = re^{i\varphi}$. Then*

$$\begin{aligned} f(z) &= \int_0^{2\pi} f(Re^{i\theta}) \mathrm{Re} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} f(Re^{i\theta}) \frac{R^2 - r^2}{R^2 - 2R\cos(\theta - \varphi) + r^2} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \mathrm{Re}\ f(Re^{i\theta}) \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \frac{d\theta}{2\pi} + iK \quad \textit{for some real constant } K. \end{aligned}$$

Proof: By Cauchy's theorem,

$$f(0) = \frac{1}{2\pi i} \int_{\mathbf{S}_R} \frac{f(\zeta)}{\zeta} d\zeta = \int_0^{2\pi} f(Re^{i\theta}) \frac{d\theta}{2\pi}.$$

Let g be the automorphism of $\overline{\mathbf{D}}(R)$ which interchanges 0 and z. Then

$$f(z) = f \circ g(0).$$

We apply the above formula to $f \circ g$. We then change variables, and use the fact that $g \circ g = \mathrm{id}$, $\zeta = g(w)$, $d\zeta = g'(w)dw$. For $R = 1$,

$$g(w) = \frac{z - w}{1 - w\bar{z}}.$$

The desired formula drops out as in the first equation. The identity

$$\mathrm{Re}\frac{Re^{i\theta} + z}{Re^{i\theta} - z} = \frac{R^2 - r^2}{R^2 - 2R\cos(\theta - \varphi) + r^2}$$

is immediate by direct computation. The third equation comes from the fact that f and the integral on the right hand side of the third equation are both analytic in z, and have the same real part, so differ by a pure imaginary constant. This concludes the proof.

For $a \in \mathbf{D}(R)$ define

$$G_R(z, a) = G_{R,a}(z) = \frac{R^2 - \bar{a}z}{R(z - a)}.$$

Then $G_{R,a}$ has precisely one pole on $\overline{\mathbf{D}}(R)$ and no zeros. We have

$$|G_{R,a}(z)| = 1 \text{ for } |z| = R.$$

Theorem 1.2. (Poisson-Jensen formula) *Let f be meromorphic non constant on $\mathbf{D}(R)$. Then for any simply connected open subset of $\mathbf{D}(R)$ not containing the zeros or poles of f, there is a real constant K such that for z in the open set we have*

$$\log f(z) = \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{Re^{i\theta} + z}{Re^{i\theta} - z}\frac{d\theta}{2\pi}$$
$$- \sum_{a \in \mathbf{D}(R)} (\mathrm{ord}_a f) \log G_R(z, a) + iK.$$

The constant K depends on a fixed determination of the logs.

Proof: Suppose first that f has no zeros or poles on $\mathbf{S}(R)$. Let

$$h(z) = f(z)\prod\left(\frac{R^2 - \overline{a}z}{R(z-a)}\right)^{\mathrm{ord}_a f}$$

where the product is taken for $a \in \mathbf{D}(R)$. Then h has no zero or pole on $\overline{\mathbf{D}}(R)$, and so $\log h$ is defined as a holomorphic function to which we can apply Theorem 1.1 to get the present formula. Then we use the fact that the log of a product is the sum of the logs plus a pure imaginary constant, on a simply connected open set, to conclude the proof in the present case.

Suppose next that f may have zeros and poles on $\mathbf{S}(R)$. Note that

$$\theta \longmapsto \log|f(Re^{i\theta})|$$

is absolutely integrable, because where there are singularities, they are like $\log|x|$ in a neighborhood of the origin $x = 0$ in elementary calculus. Let R_n be a sequence of radii having R as a limit. For R_n sufficiently close to R, the zeros and poles of f inside the disc of radius R_n are the same as the zeros and poles of f inside the disc of radius R, except for the zeros and poles lying on the circle $\mathbf{S}(R)$. The left hand side of the formula is independent of R_n. Let

$$\varphi_n(\theta) = \log|f(R_n e^{i\theta})| \qquad \text{and} \qquad \varphi(\theta) = \log|f(Re^{i\theta})|.$$

Then φ_n converges to φ. Outside small intervals around the zeros or poles of f on $\mathbf{S}(R)$, the convergence is uniform. Near the zeros and poles of f on the circle, the contribution of the integrals over small θ-intervals is small. Hence

$$\int_0^{2\pi} \log|f(R_n e^{i\theta})|\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\frac{d\theta}{2\pi} \quad \text{converges to} \int_0^{2\pi} \log|f(Re^{i\theta})|\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\frac{d\theta}{2\pi},$$

thus proving the formula in general.