RICHARD H. GALLAGHER

FINITE ELEMENT ANALYSIS

Fundamentals

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PRENTICE-HALL, INC., Englewood Cliffs, New Jersey

Library of Congress Cataloging in Publication Data

GALLAGHER, RICHARD H.
Finite element analysis: fundamentals.

(Civil engineering and engineering mechanics series)
1. Finite element method. 2. Structures, Theory of.
I. Title
TA646.G33 624'.171 74-4339
ISBN 0-13-317248-1

10 9 8 7 6 5 4 3 2 1

Printed in the United States of America

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

Preface

New fields generally progress through three periods of technological development. During the first period, the development progresses via the periodical literature and is coordinated only through the medium of occasional survey papers. Applications in practice are quite scarce. In the second phase, monograph-type texts appear and give a comprehensive view of the field for individuals actively engaged in further development of the field and applications are recorded by advanced technology groups in organizations with large resources. Finally, the applications spread to nearly all levels of practical activity and in the universities the subject matter of the field is offered as standard academic fare.

Finite element analysis has only recently emerged from the second of the phases above. A number of excellent monographic texts have appeared but a need exists for a text directed toward the conventional course offering and toward the individual with no prior acquaintance with the field. This book is intended to serve this purpose. It is oriented toward a graduate level course for students specializing in solid mechanics. This would include students enrolled in the fields of mechanical and aerospace engineering, naval architecture, engineering mechanics, and civil engineering. To the extent that a bias exists toward one of these fields it is directed toward civil engineers pointing toward structural engineering practice.

It is also hoped that this text will appeal to engineers already in practice, both to those seeking an introduction to a technology that was not found among courses of instruction during their period of formal education and to those who are routinely involved in finite element analysis and seek reference to basic proofs and formulative procedures. Much of the material contained herein in fact achieved realization in conjunction with numerous short courses offered by the author to engineers in practice.

The subject matter of this text requires some familiarity with the theory of elasticity and matrix structural analysis, and these imply an exposure to partial differential equations, the algebra of large-order equations, and the theory of structural analysis. Although it is the writer's belief that each of these topics is given adequate development from basic principles in the early chapters of the text, it is the experience of the author that the extent of coverage of finite element analysis in the usual course offering leaves very little room for a first-time assimilation of the prerequisities. We hasten to add, however, that sufficient background in theory of elasticity is normally found in the modern sophomore or junior year courses described by the title *continuum mechanics.*

The term *matrix structural analysis* requires clarification since it has been common to collect under this heading nearly all aspects of procedures related to digital computer applications in structural engineering. There has been a trend, however, toward the separation of procedures for constructing and solving the equations that describe the total structural problem, involving the connection of simple structural elements, from the formulative aspects of the elements. The former can be developed in large measure in terms of such elements as truss and framework members, whose theoretical basis can be established by only a very modest incursion into the latter area, and it is to this topic that the designation matrix structural analysis is applied.

Theoretical developments in finite element analysis have placed great dependence on the calculus of variations. We have chosen to exclude this topic from the group of prerequisite subjects since the writer has found it unrealistic to expect that students otherwise capable of initiating study in finite element analysis will have received a formal course in this topic prior to the study of finite element analysis.

The contents of this book are almost entirely devoted to the development of basic theoretical principles and, with the exception of Chapter 1, only scant attention is given to the practical application of finite element analysis. There is a wealth of such information available in the open literature, some of which can be found in the references given at the close of each chapter. In Chapter 1, in addition to portraying some representative applications of the method, we outline its developmental history, give a thumbnail sketch of the varieties of elements to be explored in later chapters of the text, and describe a motivating factor in the method, the concept of the *general-purpose* program.

Chapter 2 is devoted to the presentation of basic definitions, terminology, coordinate systems, and properties possessed by all finite element relationships independent of their mode of formulation. Chapter 3 details one method of constructing the equations of the complete structure from the equations of the individual elements—the direct stiffness method. Other methods for accomplishing this objective are outlined or mentioned in later chapters but, as we have noted, the objective of this text is to concentrate upon matters of element formulation.

Although we do not exclude one-dimensional members (e.g., axial members, beam segments) from our development, but in fact employ them liberally in the exemplification of basic theory, the prime motivation in finite element analysis is nevertheless the analysis of two- and three-dimensional continua. An understanding of the basic relationships of elasticity is therefore essential to the study of the method and we develop these from basic considerations in Chapter 4.

Two broad classifications of general procedures for formulation of element equations are covered in this text. *Direct methods*, described in Chapter 5, are appealing in their simplicity and rationale. The direct formulative process gives considerable insight into those conditions which are met by element formulation and those which are not. *Variational methods* (Chapter 6) are currently the more popular procedures for element formulation. Such methods, under well-defined circumstances, are valuable in furnishing assurances on convergence of the numerical solution and in ensuring that certain formulations yield upper or lower bounds at a given level of analysis refinement. In Chapter 6 we apply the variational methods to element formulation and in Chapter 7 we demonstrate the extension of the same ideas to the formulation of the complete structure. In this way we establish an alternative and broader view of system analysis than that given in Chapter 3.

At this point it is pertinent to take note of what the author believes to be a particular feature of this text. At the time of its preparation, all of practice and the majority of existing finite element theory dealt with finite element formulations in the class of *displacement-based* (i.e., stiffness or potential energy) procedures. Alternative formulations, based on assumed stress fields and even on both displacement and stress fields, hold considerable promise, however, and the author can foresee the possibility that all the alternatives will ultimately be on equal footing in practice. Thus, close attention is given to these alternatives in Chapters 5 through 7.

The portion of the text dealing with basic theoretical considerations concludes with Chapter 8, in which we examine procedures for the functional representation of element behavior and extend these same ideas to the representatation of element geometry. Concepts and formulations established in this chapter are probably the more generally useful than those dealt with in the prior chapters since they apply with equal force to nonstructural finite element analysis.

Specific forms of elements are given detailed examination in each of

Chapters 9 through 12. These encompass *plane stress elements* (Chapter 9), *solid elements* of general and special form (Chapters 10 and 11, respectively), and *plate flexure elements* (Chapter 12). By the same token greater emphasis is placed on references to the open literature than in the previous four chapters.

Chapter 13 treats a special form of behavior, *elastic instability*. The theory developed in this chapter applies equally well to all types of elements and for this reason it is expedient to employ once again the simplest types of elements—axial and frame members.

Three types of problems for assignment are presented. The first category comprises problems intended to exercise theoretical concepts and contains problems of the type assigned in traditional structural mechanics courses. A second category is devoted to intrinsically finite element problems but is intended for hand-calculation, e.g., the formulation of new element relationships or the solution of a structure describable by no more than three algebraic equations. Finally, we present data for problems with known classical or alternative solutions that are representable by the finite element method in terms of a relatively large number of equations. The assignment of such problems may be accomplished in many ways but one scheme the author has found effective has been to assign a different gridwork to each student in the class. Correlation of the results in class then gives the added benefit of describing the accuracy and rate of convergence of the finite element solution.

The finite element method is a technology that is wedded to digital computer analysis and it may appear surprising that no coded algorithms are presented herein. The author believes that few, if any, instructors or independent users of this text will find difficulty in obtaining access to widely distributed general-purpose finite element programs (e.g., STRUDL-II) suitable for the performance of the problem assignments discussed above. Alternatively, simpler finite element programs are found in numerous reports and texts.

It is conceivable that the subject matter of this text could be covered in the conventional three-hour per week, fifteen-week course. In the experience of the author this would require a higher proficiency in the prerequisite subjects (theory of elasticity, matrix structural analysis) and in the calculus of variations as well than would be brought to a course by the majority of students. The instructor may therefore choose to eliminate coverage of one or more of the last four chapters. In a trimester system, on the other hand, it would appear feasible to form a sequence of one ten-week course in matrix structural analysis and two succeeding ten-week finite element courses. The second finite element course would give latitude for coverage of such advanced topics as finite element theory and analysis for problems in soil mechanics, heat transfer, fluid flow, and other nonstructural applications; nonlinear problems; and the analysis of transients.

The author's thanks go out to very many former students and industrial colleagues who read various segments of the manuscript and offered both criticism and helpful suggestions. A debt of gratitude is owed to Professors J. T. Oden of the University of Texas and G. McNeice of Waterloo University for contributions to Chapter 9, and to Prof. K. Washizu of Tokyo University for his study and comments of Chapters 6 and 7. Special thanks are due to Professor Sidney Kelsey of the University of Notre Dame for his detailed review of nearly all chapters of the text and for his extensive and always perceptive comments, to Mr. James Bacci, Production Editor, Prentice-Hall and other members of the Prentice-Hall staff, including Barbara Cassel, for limitless patience in what proved to be a worldwide enterprise, and to Mrs. Helen Wheeler for her peerless typing of the manuscript and for being ever-watchful for incomplete sentences and grammatical miscues.

RICHARD H. GALLAGHER

List of Symbols

The following is a list of the principal symbols used in this text. Various other symbols are defined where they appear; this is often the case with symbols used to designate matrices (especially in Chapter 6), or which appear only in tables or figures. Symbols which have two distinctly different meanings are distinguished by subscripting (for example, L denotes length while L_i symbolizes an area or volume coordinate). Subscripts and superscripts applied to symbols with only one general meaning are not given below but are rather identified in the text where they appear.

Matrices are denoted by boldface letter within the symbols [ ] (for a rectangular matrix), { } (for a column vector) and ⌊ ⌋ (for a row vector). The definition of the boldface matrix symbol applies also to the lightface (that is, the non-boldface), subscripted, terms of the matrix. For example, the definition of the $n \times 1$ vector {**a**} applies to the individual components $a_1 \ldots a_i \ldots a_n$. The boldface symbol used to denote a matrix is often used in the non-boldface, unsubscripted form of a scalar with entirely different meaning, although in some cases the same meaning is preserved.

Overbars denote specified quantities. Primes denote differentiation.

- A Area
- [**A**] Matrix relating stresses to joint forces.
- [$\mathcal{A}$] Kinematics matrix. Coefficients of the relationships between element joint and global joint displacements.
- a Dimension

{a} Vector of parameters in assumed displacement field.
[B] Matrix relating assumed displacement field parameters to joint displacements.
[𝔅] Statics matrix. Coefficients of the relationships between element joint and global joint forces.
$b_{0_i}, b_{1_i}, b_{2_i}$ (i = 1, 2, 3). Coefficients in area coordinate equation.
C Constant in Poisson equation.
[C] Matrix relating assumed displacement field parameters to strain field.
$c_{0_i}, \ldots, c_{3_i}$ (i = 1, 2, 3, 4). Coefficients in volume coordinate equation.
D Plate flexural rigidity.
[D] Matrix relating joint displacements to strain field.
{d} Eigenvector.
d.o.f. Degree of freedom
E Elastic modulus
[E] Matrix of elastic constants
e Multiplier of variation.
{F} Vector of element nodal forces.
[𝔉] Global flexibility matrix.
{f} Element flexibility matrix.
G Shear modulus.
[G] Constraint matrix.
I Moment of inertia.
[I] Identity matrix.
$\mathcal{I}$ Value of integral.
i, j, k Dummy subscripts or superscripts.
J St. Venant torsion constant.
[J] Jacobian matrix.
[K] Global stiffness matrix.
[k] Element stiffness matrix.
L Length.
L_i Area (i = 1, 2, 3) or volume (i = 1, 2, 3, 4) coordinate.
ℓ_x, ℓ_y, ℓ_z Direction cosines.
{M} Vector of joint bending moments.
$\mathfrak{M}, M_x, M_y, M_{xy}$ General internal moment vector in plate bending (line moments) and components.
m Order of polynomial series.
[m] Element mass matrix.
$\mathfrak{N}$ Number of sides of a polygon.
[N], ⌊N⌋ Matrix of shape functions.
n Number of degrees of freedom.
[O], {O} Null matrix and vector.

{**P**}	Vector of global nodal forces.
p	Number of elements.
⌊**p**⌋	Matrix of coefficients of a polynomial series.
Q_x, Q_y	Shear line loads in plate flexure.
q	Distributed load intensity.
R	Residual.
[**R**]	Static equilibrium matrix, relating element forces to each other.
[$\mathcal{R}$]	Central matrix in generalization of one-dimensional interpolation function to two dimensions.
r	Radial coordinate; number of constraint equations.
S, S_u, S_σ	General surface and surfaces where displacements and stresses are prescribed, respectively.
[**S**]	Stress matrix, connecting joint displacements to components of stress field.
[$\mathbb{S}$]	Stress matrix, connecting joint displacements to stresses at specified points.
s	Coordinate.
{**s**}	Vector of constants in constraint equations.
T, T_x, T_y, T_z	Surface (boundary) traction vector and components.
t	Thickness.
U, U^*	Strain energy and complementary strain energy.
u	Surface (boundary) displacement vector.
u, v, w	Displacement components (in interior and of surface points).
V, V^*	Potential and complementary potential of applied loads.
vol	Volume.
W	Work.
$\mathcal{W}$	Descriptor of displacement field variation.
X, X, Y, Z	Body force vector and components.
x, y, z	Cartesian coordinates.

Greek Symbols

α	Coefficient of thermal expansion.
β	Beta function (Sect. 8.3.1).
{$\boldsymbol{\beta}$}	Vector of parameters of assumed stress field.
Γ	Gamma function (Sect. 8.3.1); warping constant (Sect. 13.3.2).
Γ_i	Cross-derivative d.o.f. [Eq. (12.31)].
[$\boldsymbol{\Gamma}$]	Transformation matrix.
{Δ}	Vector of nodal point displacements.
δ	Variational operator; infinitesmal change.

$\boldsymbol{\epsilon}$	General strain vector (includes both normal and shear strain).
$\epsilon_x, \epsilon_y, \epsilon_z$	Normal strains.
ξ, η, ζ	Nondimensional spatial coordinates.
θ	Angular displacement (angle of measure in Ch. 11).
$\boldsymbol{\kappa}, \kappa_x, \kappa_y, \kappa_{xy}$	Vector of curvatures in plate bending, and components.
$[\boldsymbol{\kappa}]$	Hessian matrix.
$\{\boldsymbol{\lambda}\}$	Vector of Lagrange multipliers.
μ	Poisson's ratio.
$\lfloor \mathbf{X}_i \rfloor$	Vector of stress field shape functions.
Π	General functional.
$\Pi_p, \Pi_p^{m_1}, \Pi_c,$	Energy functional (subscripts and superscripts denote specific type).
π	3.1416 . . .
$[\boldsymbol{\rho}]$	Material mass density matrix.
$\boldsymbol{\sigma}$	General stress field vector (includes both normal and shear stresses).
$\sigma_x, \sigma_y, \sigma_z$	Normal stresses.
$\{\boldsymbol{\sigma}\}$	Vector of node point stresses.
$\tau_{xy}, \tau_{yz}, \tau_{xz}$	Shear stresses.
Υ	Temperature change above ambient or stress-free state.
υ	Thermal conductivity.
Φ	Stress function
$\{\boldsymbol{\Phi}\}$	Vector of node point stress functions.
ϕ	Angle of measure or circumferential angular coordinate; Weighting factor in weighted residual integral.
Ω	Loading function for plate bending.
$[\boldsymbol{\Omega}]$	Mixed format force-displacement matrix.
$\{\boldsymbol{\omega}\}$	Vector of eigenvalues.

Contents

1

Introduction

In problems of structural mechanics the design analyst seeks to determine the distribution, or *field*, of stresses throughout the structure to be designed. On occasion it is necessary to calculate the displacements at certain points of the structure to ensure that specified clearances are not violated. In some cases, especially where the loads and structural response are time-dependent, it is necessary to compute the entire distribution, or field, of displacements. The calculated stress field should represent a system of internal and external forces which are everywhere in *equilibrium* and simultaneously the displacements should be continuous (the condition of *compatibility*).

In initiating the determination of a system of stresses and displacements for a given design problem, the designer must first define the governing equations of the problem that, in one form or another, stipulate the satisfaction of the conditions of equilibrium and compatibility. A basic difficulty in this regard, quite apart from the solvability of the chosen equations, is the ability of the equations to represent the design conditions. Complications in geometry, loading, and material properties enter into this consideration.

Taking note of the discrepancies likely to accrue from the sources above, the analyst moves ahead with the process of solution of the chosen equations. When the behavior being examined is of a two- or three-dimensional nature, these are of the form of partial differential equations. Rarely do exact solutions exist for such equations and only slightly more often does it prove feasible to construct an adequate solution with an approximation consist-