Longman Mathematical Texts

A short course in
General Relativity

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Preface

This book provides a short course in general relativity, intended primarily for senior undergraduates or first-year graduate students in physics, mathematics, or related subjects such as astrophysics. Our intention was to produce a book suitable for those who may only take one course in the subject, typically of one or two terms, or one semester duration, but it should also serve as an introduction to the excellent and more comprehensive texts which have appeared in recent years.

Most students approaching the subject require an introduction to tensors, which provide the language of relativity, and these are dealt with in Chapter 1 and the first half of Chapter 2. The latter half of Chapter 2 discusses the geodesic equations, Chapter 3 the field equations, and Chapter 4 applies the results learned to physics in the vicinity of a massive object. Throughout we have tried to compare new results with their Newtonian counterparts. Chapters 5 and 6, on gravitational radiation and cosmology respectively, give further applications of the theory, but students wanting a more detailed knowledge of these topics (and indeed all topics) would have to turn to the longer texts referred to in the body of the book. We finish with an appendix, where special relativity is reviewed, and presented in a form which makes contacts with the general theory easier to establish. Chapters 5 and 6 are independent, and either or both could be omitted to produce a shorter course. Exercises have been provided at the ends of most sections, and problems at the ends of chapters. The former are quite often straightforward (but possibly tedious) verifications needed for a first reading of the book, while the latter are conceivably not so necessary.

General relativity is becoming much more of an experimental subject, and so that the reader may savour something of the way modern technology is brought to bear on the problems which beset experimenters, we have given references (mainly to periodical articles) where appropriate.

Independently, and quite coincidentally, much of the material here has been taught during the past decade at both New Paltz and Sussex, and its consolidation into a single text was the result of one of us taking his sabbatical leave in England. While responsibility for errors is entirely the authors' we would like to mention with gratitude Bob Marchini, John McNamara, John Ray, Eric Shugart and Stacie Swingle, all of whom have been of assistance in one way or another, and not least our wives. Early work on the book owes much to the help and encouragement of Arlene Nightingale, while the final typescript was accurately produced at extremely short notice by Jill Foster. Finally, we would like to thank Professor Alan Jeffrey for his encouragement and the staff at Longman for their courteous cooperation.

Introduction

The originator of the general theory of relativity was Einstein, and in 1919 he wrote [1]: *The special theory, on which the general theory rests, applies to all physical phenomena with the exception of gravitation; the general theory provides the law of gravitation and its relation to the other forces of nature.* The claim that the general theory provides the law of gravitation does not mean that H. G. Wells' Mr Cavor could now introduce an antigravity material and glide up to the Moon, nor, for example, that we might produce intense permanent gravitational fields in the laboratory, as we can electric fields. It means only that all the properties of gravity of which we are aware are explicable by the theory, and that gravity is essentially a matter of geometry. Before saying how we get to the general from the special theory, we must first discuss the principle of equivalence.

In electrostatics, when a test particle of charge q and inertial mass m_i is placed in a static field $\mathbf{E}$, it experiences a force $q\mathbf{E}$, and undergoes an acceleration

$$\mathbf{a} = (q/m_i)\mathbf{E}. \tag{I.1}$$

In contrast, a test particle of gravitational mass m_g and inertial mass m_i placed in a gravitational field $\mathbf{g}$ experiences a force $m_g\mathbf{g}$, and undergoes an acceleration

$$\mathbf{a} = (m_g/m_i)\mathbf{g}. \tag{I.2}$$

It is an experimental fact (known since Galileo's time) that different particles placed in the same gravitational field acquire the same acceleration (see Fig. I.1(a)). This implies that the ratio m_g/m_i appearing in equation (I.2) is the *same* for all particles, and by an appropriate choice of units this ratio may be taken to be unity. This equivalence of gravitational and inertial mass (which allows us to drop the qualification, and simply refer to *mass*) has been checked experimentally by Eötvös (in 1889 and 1922), and more recently and more accurately (to one part in

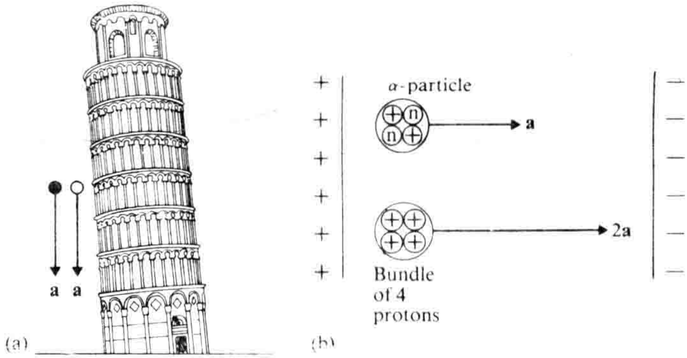

Fig. I.1 Test particles in (a) a gravitational field, and (b) an electrostatic field.

10^{11}) by Dicke and his coworkers (in the 1960s). In contrast, the ratio q/m_i occurring in equation (I.1) is not the same for all particles (see Fig. I.1(b)).

With these preliminaries over, we may now consider the principle of equivalence, and it is instructive to do so from the point of view of Einstein's freely falling elevator. If we consider a projectile shot from one side of the elevator cabin to the other, the projectile appears to go in a straight line (the elevator cable being cut) rather than in the usual curved trajectory. Projectiles that are released from rest relative to the cabin remain floating weightless in the cabin. Of course, if the cabin is left to fall for a long time, the particles gradually draw closer together, since they are falling down radial lines towards a common point which is the centre of the Earth. However, if we make the proviso that the cabin is in this state for a short time, as well as being spatially small enough for the neglect of tidal forces in general, then the freely falling cabin (which may have X, Y, Z-coordinates chalked on its walls, as well as a cabin clock measuring time T) looks remarkably like an inertial frame of reference, and therefore the laws of special relativity hold sway inside the cabin. (The cabin must not only occupy a small region of spacetime, it must also be non-rotating with respect to distant matter in the universe [2].) All this follows from the fact that the acceleration of any particle relative to the cabin is zero because they both have the same acceleration relative to the Earth, and we see that the equivalence of inertial and gravitational mass is an essential feature of the discussion. We may incorporate these ideas into the *principle of equivalence*, which is this: *In a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are the laws of special relativity* [3].

As a result of the above discussion, the reader should not believe that we can actually transform gravity away by turning to a freely falling reference frame. It is absolutely impossible to transform away a permanent gravitational field of the type associated with a star (as we shall see in Chapter 3), but it is possible to get closer and closer to an ideal inertial reference frame if we make our laboratory occupy smaller and smaller regions of spacetime.

The way in which Einstein generalised the special theory so as to incorporate gravitation was extremely ingenious, and without precedent in the history of science. Gravity was no longer to be regarded as a force, but as a manifestation of the curvature of spacetime itself. The new theory, known as the *general theory of relativity* (or *general relativity* for short), yields the special theory as an approximation in exactly the way the principle of equivalence requires. Because of the curvature of spacetime, it cannot be formulated in terms of coordinate systems based on inertial frames, as the special theory can, and we therefore use arbitrary coordinate systems. Indeed, global inertial frames can no longer be defined, the nearest we can get to them being freely falling non-rotating frames valid in limited regions of spacetime only. A full explanation of what is involved is given in Chapter 2, but we can give a limited preview here.

In special relativity, the invariant expression which defines the proper time τ is given by

$$c^2\, d\tau^2 = \eta_{\mu\nu}\, dX^\mu\, dX^\nu, \tag{I.3}$$

where the four coordinates X^0, X^1, X^2, X^3 are given in terms of the usual coordinates T, X, Y, Z by

$$X^0 \equiv cT, \qquad X^1 \equiv X, \qquad X^2 \equiv Y, \qquad X^3 \equiv Z. \tag{I.4}$$

(See Section A.0, but note the change to capital letters. See also Section 1.1 for an explanation of the summation convention.) If we change to arbitrary coordinates x^μ, which may be defined in terms of the X^μ in any way whatsoever (they may, for example, be linked to an accelerating or rotating frame), then the expression (I.3) takes the form

$$c^2\,\mathrm{d}\tau^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \tag{I.5}$$

where

$$g_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial X^\rho}{\partial x^\mu}\frac{\partial X^\sigma}{\partial x^\nu}.$$

This follows from the fact that $\mathrm{d}X^\rho = (\partial X^\rho/\partial x^\mu)\,\mathrm{d}x^\mu$. In terms of the coordinates X^μ, the equation of motion of a free particle is

$$\mathrm{d}^2X^\mu/\mathrm{d}\tau^2 = 0, \tag{I.6}$$

which, in terms of the arbitrary coordinates, becomes

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0, \tag{I.7}$$

where

$$\Gamma^\mu_{\nu\sigma} = \frac{\partial x^\mu}{\partial X^\rho}\frac{\partial^2 X^\rho}{\partial x^\nu \partial x^\sigma},$$

as a short calculation (involving the chain rule) shows. Einstein's proposals for the general theory were that in any coordinate system the proper time should be given by an expression of the form (I.5), and that the equation of motion of a free particle (i.e. one moving under the influence of gravity alone, gravity no longer being a force) should be given by an expression of the form (I.7), but that (in contrast to the spacetime of special relativity) *there are no preferred coordinates* X^μ which will reduce these to the forms (I.3) and (I.6). This is the essential difference between the spacetimes of special and general relativity. The curvature of spacetime (and therefore gravity) is carried by the $g_{\mu\nu}$, and as we shall see, there is a sense in which these quantities may be regarded as gravitational potentials. We shall also see that the $\Gamma^\mu_{\nu\sigma}$ are determined by the $g_{\mu\nu}$, and that it is always possible to introduce *local* inertial coordinate systems of *limited extent* in which $g_{\mu\nu} \simeq \eta_{\mu\nu}$ and $\Gamma^\mu_{\nu\sigma} \simeq 0$, so that equations (I.3) and (I.6) hold as approximations. We thus recover special relativity as an approximation, and in a way which ties in with our discussion of the principle of equivalence.

Because the introduction of curvature forces us to use arbitrary coordinate systems, we need to formulate the theory in a way which is valid in all coordinate systems. This we do by using tensors, the necessary algebra for which is developed in Chapter 1; the way these fit into the theory is explained in Chapter 2. It might be thought that this arbitrariness causes problems, because the coordinates lose the simple physical meanings that the preferred coordinates X^μ of special relativity have. However, we still have contact with the special theory at the local level, and in this way problems of physical meaning and the correct formulation of equations may be overcome. The basic idea is contained in the *principle of general covariance*, which may be stated as follows: *A physical equation of general relativity is generally true in all coordinate systems if (a) the equation is a tensor equation (i.e. it preserves its form under general coordinate transformations), and (b) the equation is true in special relativity.* The way in which this principle works, and the reason why it works, are explained in Section 2.6.

General relativity should not only reduce to special relativity in the appropriate limit, it should also yield Newtonian gravitation as an approximation. Contacts and comparisons with Newtonian theory are made in Sections 2.7, 2.8, 2.9 and 2.10, and extensively in Chapter 4, where we discuss physics in the vicinity of a massive object. These reveal differences between the two theories which provide possible experimental tests of the general theory, and for convenience we list here the experimental and observational evidence concerning these tests, the so-called five tests of general relativity.

1. *Perihelion advance.* General relativity predicts an anomalous advance of the perihelion of planetary orbits. The following (and many more) observations exist for the solar system [4]:

Mercury	$43.11 \pm 0.45''$	per century,
Venus	$8.4 \pm 4.8''$	per century,
Earth	$5.0 \pm 1.2''$	per century.

The predicted values are 43.03″, 8.6″ and 3.8″ respectively.

2. *Deflection of light.* General relativity predicts that light deviates from rectilinear motion near massive objects. The following (and many more) observed deflections exist for light passing the Sun at grazing incidence:

1919	Greenwich Observatory	$1.98 \pm 0.16''$,

1922	Lick Observatory	$1.82 \pm 0.20''$,
1947	Yerkes Observatory	$2.01 \pm 0.27''$,
1972	Mullard Radio Astronomy Observatory, Cambridge (using radio sources and interferometers)	$1.82 \pm 0.14''$.

The predicted value is $1.75''$.

3. *Spectral shift.* General relativity predicts that light emanating from near a massive object is red-shifted, while light falling towards a massive object is blue-shifted. Numerous observations of the spectra of white dwarfs, as well as the remarkable terrestrial experiments carried out at the Jefferson Laboratory [5] verify the general-relativistic prediction.

4. *Time delay in radar sounding.* General relativity predicts a time delay in radar sounding due to the gravitational field of a massive object. Experiments involving the radar sounding of Venus, Mercury and the spacecrafts *Mariner* 6 and 7, performed in the 1960s and 1970s, have yielded agreement with the predicted values to well within the experimental uncertainties [6].

5. *Geodesic effect.* General relativity predicts that the axis of a gyroscope which is freely orbiting a massive object should precess. For a gyroscope in a near-Earth orbit this precession amounts to $8''$ per year, and an experiment involving a gyroscope in an orbiting satellite is (at the time of writing) being prepared [7].

Finally, let us say something about the notation used in this book. Wherever possible we have chosen it to coincide with that of the more recent and influential texts on general relativity. Greek suffixes (μ, ν, etc.) have the range 0, 1, 2, 3, while English suffixes from the middle of the alphabet (i, j, etc.) have the range 1, 2, 3. The signature of the metric tensor is -2, i.e. $\eta_{00}=1$, $\eta_{11}=\eta_{22}=\eta_{33}=-1$. Rather than use gravitational units in which the gravitational constant G and the speed of light c are unity, we have retained G and c throughout, expect in Chapter 6 where $c=1$. In the sections on tensor algebra, tensor analysis and curvature, the underlying space or manifold is of arbitrary dimension, and we have used English suffixes from the beginning of the alphabet (a, b, etc.) to denote the arbitrary range $1, 2, \ldots, N$.

Where an equation defines some quantity or operation, the symbol ≡ is used on its first occurrence, and occasionally thereafter as a reminder. Important equations are displayed between parallel lines.

Notes

1. *The Times*, London, 28 November 1919.
2. This statement is related to *Mach's principle*. For a discussion, see Weinberg, 1972, §3.
3. Some authors distinguish between weak and strong equivalence. Our statement is the strong statement; the weak one refers to freely falling particles only, and not to the whole of physics.
4. The figures are taken from Duncombe, 1956.
5. See Pound and Rebka, 1960.
6. See Shapiro, 1968; Shapiro *et al.*, 1971; and Anderson *et al.*, 1975.
7. See the paper by Everitt, Fairbank and Hamilton in Carmeli *et al.*, 1970.

Contents

1

Vectors and tensors

1.0 Introduction

In this first chapter we deal with some algebraic preliminaries, namely the concepts of vector spaces, their duals, and spaces which may be derived from these by the process of tensor multiplication. The treatment is quite general, though restricted to real finite-dimensional vector spaces, and we pay particular attention to the effect of changes of bases on components, since this aspect is of particular importance when the ideas developed here are applied to general relativity in subsequent chapters.

1.1 Vector spaces

We shall not attempt a formal definition of a vector space, but assume that the reader has some familiarity with the concept. The excellent text by Halmos is a suitable introduction for those new to the concept [1].

The essential features of a vector space are that it is a set of *vectors* on which are defined two operations, namely addition of vectors and the multiplication of vectors by *scalars*; that there is a zero vector in the space; and that each vector in the space has an inverse such that the sum of a vector and its inverse equals the zero vector. It may be helpful to picture the set of vectors comprising a vector space as a set of arrows emanating from some origin, with addition of vectors given by the usual parallelogram law, and multiplication of a vector by a scalar as a scaling operation which changes its length but not its direction, with the proviso that if the scalar is negative, then the scaled vector will lie in the same line as the original, but point in the opposite direction. In this picture the zero vector is simply the point which is the origin (an arrow of zero length), and the inverse of a given vector is one of the same length and in the same line as the given vector, but pointing in the opposite direction.

In this text our scalars are real numbers, so our vector spaces are termed real. More generally, the scalars could be taken from any field F, giving a vector space over F. We denote the field of real numbers by $\mathbb{R}$, and to distinguish vectors from scalars the former will be printed in bold-faced type ($\mathbf{0}$, $\mathbf{e}$, $\mathbf{v}$, $\boldsymbol{\lambda}$, etc.).

The notion of linear independence is of central importance in vector-space theory. If, for any scalars $\lambda^1, \ldots, \lambda^K$,

$$\lambda^1\mathbf{v}_1 + \lambda^2\mathbf{v}_2 + \cdots + \lambda^K\mathbf{v}_K = \mathbf{0} \tag{1.1.1}$$

implies that $\lambda^1 = \lambda^2 = \cdots = \lambda^K = 0$, then the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ is said to be *linearly independent*. A set of vectors which is not linearly independent is *linearly dependent*. Thus for a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_K\}$ there exists a non-trivial linear combination of the vectors which equals the zero vector. That is, there exist scalars $\lambda^1, \ldots, \lambda^K$, not all zero (though some may be) such that

$$\lambda^1\mathbf{v}_1 + \lambda^2\mathbf{v}_2 + \cdots + \lambda^K\mathbf{v}_K = \mathbf{0}. \tag{1.1.2}$$

It is appropriate here to say something about notation. In the above we have labelled the vectors with a subscript and the scalars with a superscript. At first sight it may be thought that the use of a superscript will lead to confusion with powers, but since we are mainly concerned with linear properties, powers seldom arise; when they do, bracketing removes any ambiguity, e.g. $(\lambda^1)^2$ denotes the square of λ^1. As we shall see, the use of subscripts and superscripts leads to a remarkably efficient notation, the efficiency of which is further improved by *Einstein's summation convention*. This is that if in any expression the same letter appears as a superscript and also as a subscript, then summation over all possible values of the letter is implied. For example, the linear combination

$$\lambda^1\mathbf{v}_1 + \cdots + \lambda^K\mathbf{v}_K = \sum_{a=1}^{K} \lambda^a\mathbf{v}_a$$

appearing above is written simply as $\lambda^a\mathbf{v}_a$, the range of summation from $a = 1$ to $a = K$ being gleaned from the context. If in any expression we have more than one range of summation, then distinctions may be made by using different alphabets (or different parts of the same alphabet) for different ranges. We shall in general use small English letters a, b, c, etc., to denote a general range from 1 to N, and small Greek letters μ, ν, σ, etc., to