Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Nonstandard Analysis

A Practical Guide with Applications

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Dialogue

On the University's premises,

- Are you familiar with the non-standard methods ?
- ???
- Non-standard Analysis, if you prefer
- I've heard about ... Infinitesimals and the like, that seem to be coming back into fashion. But why call it a method ?
- Because it's not so much a question of bringing infinitesimals into fashion, but rather of furnishing a new proving tool, a non-standard one for those unfamiliar with it.
- Do you intend to modify our standards of reasoning ? You would'nt be the first... But old math is good enough for me !
- Not at all ! The non-standard method introduces intermediate objects - infinitesimals, for instance - by means of a very simple trick of language, in order to simplify the proofs - mainly whenever asymptotic behaviours are concerned - and to make them close to heuristic approaches.
- But you add something ! It's no longer the same math ! Enlarging the frame so as to introduce everything you want ... no wonder that your reasoning becomes simpler ! But is it still valid ?
- Undoubtedly ! Non-standard mathematics is strictly equivalent to the standard one : every non classical reasoning about the usual mathematical objects is equivalent to a classical one.
- So, there is nothing new in your method ! Because if you have a non classical proof, there is also a classical one. I prefer to search for the latter. Your method is pointless : either you prove known results or you produce a new result that you could, after all, prove in another way. What's the gain ?
- Your are one of those who prefer to march up towards the source of the river to get accross, instead of using a bridge down-stream. Because that is what it amounts to : to use bridges in order to avoid circuitous paths.

Recall Leibniz : "On ne diffère du style d'Archimède que dans les expressions, qui sont plus directes dans notre méthode et plus conformes à l'art d'inventer...".

Replace "Archimède" by "Bourbaki" and everything is said.

- Well ! But I heard that your method is quite complicated ; a logician's affair with languages, models and all that stuff... It seems that Robinson's book begins with fifty pages on logic in order to justify this infinitesimal nonsense !
- Don't worry about those pages ! They are conceived for the specialist's sake, to make the foundations irrefutable ; but you, as a mathematician, you should read the sequel, with all its promising developments.
- I'll try ; but really, I'm not convinced that non-standard methods may seriously change the mathematical landscape. Apart from esthetical and historical aspect, where's the importance for non-specialists ?
- True, if you only use Non-Standard Analysis to get easier presentations of well-known classical topics. But the main interest is elsewhere, in the fascinating world of applied mathematics ! For instance, engineers have to master a lot of perturbation phenomena for which classical tools are rather hard to work out. Due to its new intermedia - like infinitesimals - and powerful principles, Non-Standard Analysis allows deep investigation of perturbations in quite a natural way...
- I'm ready to agree, if you provide some examples. Meanwhile, I wonder by which miracle new objects may be introduced without further changes. Most likely these objects were already present, but they couldn't be talked about.
- Right ! Usually, to introduce an object, one begins with a definition ; what is non-standard is to introduce undefined objects, together with suitable restrictions.

For instance, the statement "there exists an element ω in the set of integers, larger than any integer which may be constructed by means of "adding a stroke", defies well established beliefs. For everybody, N is the collection I, II, III, etc...

However, we may consistently use such an undefined ω, subject just to the conditions $\omega > I$, $\omega > II$, $\omega > III$, etc... ; indeed, in a proof involving ω, one only uses a finite number of the above conditions ; hence the argument is still valid if one replaces ω by some "genuine" large enough integer.

Thus, ω doesn't disturb arithmetic and it deserves to be called infinitely large, as do ω - I, ω - II, ω - III or ω + I, ω + II, ω + III, ...

Similarly, the Non-Standard method introduces the undefined predicate "standard" with some restrictions in order to use it without betraying classical mathematics.

- Now I'm really lost ! What's this swindle about the set of integers ? Never heard such a nonsense !!

In the shadow of a grove,

Lutz : Here's a gentleman who is ready for our yellow booklet !

Goze : Indeed, but now he is angry about this story with ω ...

Lutz : Don't worry, it's only the initial shock necessary to get a non-standard mind ! he'll survive ...

Goze : Hey ! look who is coming there ! It's Georges with the catch-word ...

Exeunt omnes.

The authors wish to thank all those who have had a beneficent influence on the shape of this book, in particular

- all those non-standard minded people at Strasbourg - Oran - Mulhouse whose works are basic references to section IV ;
- our friends Dr. Tewfik SARI whose everyday collaboration was inestimable and Dr. Wilfried REYES whose linguistic improvements made our "formal" english text close to an actual one ;
- Prof. W.T. VAN EST and Prof. E.M. de JAGER whose kind invitation to Amsterdam and repeated encouragement strongly stimulated our developments on asymptotics ;
- Miss Huguette HAUSHALTER who typed the text with care and high efficiency

Of course, it is not possible to express our gratitude to Prof. Georges REEB in a few words. A few years ago, as he claimed in every corridor that Non-Standard Analysis was "something new" that should be a worthy tool in perturbation problems ; we believed it and thus had to write a book on it...

That he is so completely in the right and so interested in developing non-standard methods is not surprising : his intuition is legendary and he has the odd belief that mathematical research should be a pleasant adventure !

Mulhouse, Leimbach, Rammersmatt,

March 1981

One aim of this work is to stimulate a large discussion within the mathematical community about the efficiency of non-standard Analysis as a tool for mathematicians.

Therefore the authors invite the readers to send them their remarks (negative or positive...) both on the subject itself and on the topics involved in the present book.

R. Lutz and M. Goze

A Christianne et Ginette

et à nos enfants

Emmanuelle

Joelle

Yannik

Christine

Emmanuelle

"The widely held belief that one cannot
get something for nothing is a superstition"
E. NELSON, BAMS 83 (1977), p. 1184.

Fîhr nix, ebb's ; diss gibbt's.

FOREWORD

The dream of an infinitesimal calculus worthy of the name, that is to say in which dx and dy are infinitesimal numbers, $\int_a^b f(x)\,dx$ is a genuine sum of such numbers, limits are attained (or almost), formulae of type

$$\varphi(x) = \int_0^x f(\xi\,,\varphi(\xi-\tau)\;d\xi\ ,$$

with τ infinitely small furnish the solution of $y' = f(x\,,y)$ that satisfies $\varphi(0) = 0$ (*) , has always been dreamed by mathematicians and such a dream deserves perhaps an epistemological inquiry.

Some other dreams, lesser maybe if compared with the achievements of calculus, have haunted the mathematician's imagination and wishful thought :

it is the idea of a world where integers can be classified as "large", "small" or even "indeterminate" without the loss of consistent reasoning, satisfy the induction principle and where the successors of small integers would remain small (**) ; a world where concrete collections, fuzzy perhaps but anyhow not finite, could be gathered in a single finite set ; a world where continuous functions would be approximated almost perfectly by polynomials of a fixed degree. In such a world, the finite realms could be explored either through the telescope or through the magnifying glass in order to gather entirely new pictures. Within such a world, the criteria of rigor set forth by Weierstrass and Göttingen, interpreted in a two-fold sense would allow for phantasy and metaphor.

This foreword is an opportunity to set forth the following remarks :

(*) This list may be extended: where $\frac{1}{\sqrt{2\pi\sigma}}\exp(-\frac{x^2}{2\sigma^2})$ for $\sigma > 0$ and σ infinitesimal would serve as a Dirac function, where the teratology of the solutions of $y' = f(x\,,y)$, with continuous, non Lipschitz f could be viewed as the regularity, seen through some appropriate glass, of the case in which f is analytic...

a) The outstanding work of A. Robinson on Non-Standard Analysis provides an astonishingly easy answer to this wishful dream. His book is still remarkable for the examples chosen in various fields of Mathematics or theoretical physics ; these examples were scarcely noticed, for questions of foundations seemed to be much more important.

b) The present work has a peculiar flavour among the books on Non-Standard Analysis published. R. Lutz and M. Goze's developments, centered around the idea of perturbation, singular perturbation and deformation, may show how the situation has evolued ever since ; although their book is research oriented, readers should acquire a good working knowledge of Non-Standard Analysis.

c) Developments arising from Non-Standard Analysis won't be utterly surprising to those willing to subscribe to the following simplified version of Brouwer, Skolem and Gödel :

Concrete sets in formalized mathematics do not cope with those provided by formalization.

d) Although the various pieces entering in c) are known since the twenties, mathematicians were not convinced that they could gather valuable results on the basis of c).

If we return to the quotation from Nelson that we have chosen as a heading, one may wonder why man disregards the use of such free gifts, whose very existence is undeniable.

Georges H. REEB

March 1981

(**) In chemistry for instance, ratios p/q with "small" integers p , q used to be considered. A distinguished mathematician gives a pleasant example :
"There should be a finite chain linking some monkey to Darwin, respecting the rules : a monkey's son is a monkey, the father of a man is a man."
Other examples could be found in the domain proper to "linguistics".

READING GUIDE

This book is intended to enable the reader to use Non Standard Analysis by himself without fear, at any level of mathematical practice, from undergraduate analysis to important research areas.

It is divided into four sections with complementary purposes.

In Section I, the concept of enlargement with transfer and idealisation properties is introduced gradually and used to prove some statements on elementary calculus. To avoid a formal non motivated definition, we surround this concept with a progressive "order of procedure" as a hand rail.

In Section II, after a quick survey of set theory and some disturbing remarks about the gap between the potential collection of "natural" integers and the formal set $\mathbb{N}$, enlargements are justified as by-products of the axiom of choice.

This study leads to a description of internal set theory (I.S.T.), an axiomatic approach to Non-Standard Analysis, which provides our game with pleasant rules. The existence of enlargements is closely related to the consistency proof of I.S.T., for it provides models of this theory.

Both approaches - with enlargements or within I.S.T. - are equivalent as regards mathematical praxis, and section III begins with a comparison of both working on general topology. For practical reasons, we go on using I.S.T., after a very small improvement to allow external sets in the discourse.

The remaining lessons of section III are devoted to a non-standard treatment of some important chapters of topology and differential calculus. At this point, the reader should be able to use N.S.A. in whatever areas of mathematical research, in which it may be efficient.

Section IV is intended to give some recent examples of such attempts about various perturbation problems in algebra and differential equations ; here N.S.A. appears as an important tool in applied mathematics, according to the original aim of Abraham Robinson. No familiarity with classical perturbation theories is required, but some insight into the literature would make comparisons possible.

Section IV begins with a check-list of what is necessary to work in it without any knowledge of I, II, III. Of course, if you are not in a hurry, it is better to start with reading lesson 0, Section I.

The style of the book is rather non linear. Every lesson - some readers may wonder at this old-fashioned word, but we like its flavour - is focused on an essential information ; various comments, including proofs if necessary, invite the reader to ponder over this information in the light of his growing non-standard knowledge.

Some exercises sprinkle the text and there are topics to be developed.

A small glossary collects the terms which play some part in the book.

One aim of this book is to stimulate a large debate among mathematicians about the use of non-standard Analysis in the current research. Therefore we heartily invite the readers to send us their reactions (even bad ones, of course...) or tell us their own experiences with N.S.A..

C O N T E N T S

IV. NON STANDARD ANALYSIS AS A TOOL IN PERTURBATION PROBLEMS

PART I : ELEMENTARY PRACTICE OF NON STANDARD ANALYSIS

Lesson 0

(quite classic)

INFINITESIMALS

Let K be a commutative totally ordered non archimedean field, containing $\mathbb{R}$ as an ordered subfield. To K we associate :

i) the ring F of finite elements ; $\mathbb{R}$ is a subring of F .

ii) the set I of infinitesimals ; I is an ideal of F and $\mathbb{R} \cap I = \{0\}$.

iii) the set of infinitely large elements : that is $K - F$.

iv) the equivalence relation $\sim$ (read "infinitely near") on K .

v) a natural injection of ordered rings $\Phi : \mathbb{R} \longrightarrow F/I$.

THEOREM. Φ <u>is an isomorphism</u>, <u>that is</u> $F = \mathbb{R} \oplus I$, <u>or finite elements are infinitely near to elements of</u> $\mathbb{R}$.

<u>Comments</u>. 1) In a non archimedean field "the tortoise cannot overtake any hare", because there is an ω larger than every integer. Such odd fields exist ; the smallest one which contains $\mathbb{R}$ is $\mathbb{R}(X)$, the field of rational fractions endowed with the degree relation.

2) As $\mathbb{R}$ is archimedean, ω is not only larger than every integer, but also larger than every element of $\mathbb{R}$. Call <u>infinitely large</u> those elements of K whose absolute value has this property. All other elements are called <u>finite</u> : each of them is bounded by a real number. The inverses of infinitely large elements together with 0 are called <u>infinitesimals</u>. Sentences i) and ii) summarize the computing rules on infinitesimals ; they imply "(infinitely large) $\times$ (non infinitesimal) = (infinitely large)". Of course, a product of type "(infinitely large) $\times$ (infinitesimal)" may take any value.

3) Define $\sim$ by " $x \sim y$ if and only if $x - y \in F$ ". This equivalence relation is compatible with addition, but not with multiplication, for I is not an ideal of K (only of F).

4) Properties i) to v) are obvious. Let us prove the theorem :

If a is a finite element of K , the set $E = \{x \in \mathbb{R} , x \leq a\}$ is bounded from above ; its least upper bound ($\mathbb{R}$ is complete !) is infinitely near to a .

Thus we have an "infinitesimal calculus", but what can we do with it ?

Lesson 1

(with a slight non standard flavour)

LIMITS

THEOREM. " $u_n \longrightarrow 0$ as $n \longrightarrow \infty$ " is equivalent to "for every infinitely large n , u_n is infinitesimal".

Comments. 1) Within $\mathbb{R}$, the second part of this sentence obviously has no sense : there are no infinitesimals but 0 , if we agree with ABEL's definition. For this reason, the infinitesimal point of view in analysis did not survive, except as a figure of speech, after CAUCHY and WEIERSTRASS replaced it - with regret - by the wellknown " $\varepsilon - \delta$ " concept of limit.

2) In the frame of Lesson 0, the sentence makes sense, but, if we consider u_n as a real sequence, we have first to select a set $\widetilde{\mathbb{N}}$ of "generalized integers" in the non euclidean field K , which contains $\mathbb{N}$ and also infinitely large elements ; then we have to extend the mapping $u : \mathbb{N} \longrightarrow \mathbb{R}$ to a mapping $\tilde{u} : \widetilde{\mathbb{N}} \longrightarrow K$ (which in the sentence above is improperly named u).

3) Take for instance $\tilde{u}_n = u_n$ if $n \in \mathbb{N}$ and $\tilde{u}_n = 0$ if $n \in \tilde{\mathbb{N}}$ is infinitely large. Then it follows from the theorem that every real sequence has limit 0 . This would considerably simplify real analysis, wouldn't it ?

4) Thus, to give our theorem some chance to be true, we have to relate closely the properties of $\tilde{u}$ with those of u .

Let us outline a proof based on such a demand. For the direct part, consider $\varepsilon > 0$, $\varepsilon \in R$. Then there is an $n_o \in \mathbb{N}$ such that the map u has the property " $\forall\, n > n_o$, $n \in \mathbb{N} \Longrightarrow |u_n| < \varepsilon$" . Suppose that the related property is true for $\tilde{u}$, that is " $\forall\, n > n_o$, $n \in \tilde{\mathbb{N}}$, $|\tilde{u}_n| < \varepsilon$ ". Thus, for every infinitely large n , we have $|\tilde{u}_n| < \varepsilon$, which implies $|\tilde{u}_n| \sim 0$.

Conversely, suppose $|\tilde{u}_n| \sim 0$ for every infinitely large $n \in \tilde{\mathbb{N}}$. Then the property " $\forall\, n > n_o$, $n \in \tilde{\mathbb{N}} \Longrightarrow |\tilde{u}_n| < \varepsilon$ " is true for every $\varepsilon > 0$, $\varepsilon \in R$ and n_o infinitely large. Hence, for every fixed ε , the statement " $\exists\, n_o$, $\forall\, n > n_o$, $n \in \tilde{\mathbb{N}} \Longrightarrow |\tilde{u}_n| < \varepsilon$ " is true. If the relation between u and $\tilde{u}$ is sufficient, we may hope that it remains true if we replace $(\tilde{u}, \tilde{\mathbb{N}})$ by $(u, \mathbb{N})$. This would end the proof.

Note the correct formulation of the theorem :

THEOREM. " $u_n \longrightarrow 0$ as $n \longrightarrow \infty$ ", <u>where</u> u_n <u>is a real sequence</u>, <u>is equivalent to "for every infinitely large</u> $n \in \tilde{\mathbb{N}}$, $\tilde{u}_n$ <u>is infinitesimal in</u> K ".

5) Consider for instance the sequence $u_n = \frac{1}{n}$. We have no information about $\tilde{u}_n$ for infinitely large n . But we certainly would ask for an $\tilde{u}$ which extends the property " $nu_n = 1$ ", that is to take $\tilde{u}_n = \frac{1}{n}$. Then our characterization of limits leads to $\frac{1}{n} \longrightarrow 0$ (of course, this is a rather complicated way to prove it...).

Our business is to find K , $\tilde{\mathbb{N}}$, and an extension rule $u \longrightarrow \tilde{u}$ so that all this works. Fortunately, we have an answer (within our classical mathematics !). Its key word is <u>enlargements</u> and it was A. ROBINSON's idea to build on this concept a new procedure in Analysis. Little by little, we