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Bases in Banach Spaces II

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Preface

Since the appearance, in 1970, of Vol. I of the present monograph [370], the theory of bases in Banach spaces has developed substantially. Therefore, the present volume contains only Ch. III of the monograph, instead of Ch. III, IV and V, as was planned initially (cp. the table of contents of Vol. I). Since this volume is a continuation of Vol. I of the same monograph, we shall refer to the results of Vol. I directly as results of Ch. I or Ch. II (without specifying Vol. I). On the other hand, sometimes we shall also mention that certain results will be considered in Vol. III (Ch. IV, V).

In spite of the many new advances made in this field, the statement in the Preface to Vol. I, that "the existing books on functional analysis contain only a few results on bases", remains still valid, with the exception of the recent book [248 a] of J. Lindenstrauss and L. Tzafriri. Since we have learned about [248 a] only in 1978, in this volume there are only references to previous works, instead of [248 a]; however, this will cause no inconvenience, since the intersection of the present volume with [248 a] is very small. Let us also mention the appearance, since 1970, of some survey papers on bases in Banach spaces (V. D. Milman [287], [288], C. W. McArthur [275] M. I. Kadec [204], § 3 and others).

The fact that the basis problem (and even the approximation problem) was solved in the negative, by Per Enflo [99] (due to its special importance, we shall present this result in § 0), has increased the usefulness of a monograph on bases and their generalizations in Banach spaces.

This volume attempts to present the results known today on generalizations of bases in Banach spaces and some unsolved problems concerning them. The style is, deliberately, the same as that of Vol. I, except that the section of Notes and remarks and the Bibliography are larger. The works which have appeared after the main part of this volume had been completed, are usually encompassed by the Notes and remarks; also, this section is more detailed, since for most of the results we indicate the paper and the place in it, where the result occurs.

We hope tnat, similarly to Vol. I, the present volume will be useful to specialists, stimulating further research, and to a large circle of those who want to use it for applications to other problems (for example, orthogonal series, summability, functional equations, etc.). Also, in order to make the book suitable for study, the necessary tools from functional analysis have been carefully explained (either by giving them, as lemmas, with their proofs, or by giving references to treatises containing their proofs). Some of our unpublished results and remarks have been also included, without any special mention.

In order to give some applications of bases and their generalizations to the study of Banach spaces, we shall publish, hopefully soon after the appearance of this volume, a part of Ch. IV of the present monograph, in the Lecture Notes in Mathematics series of the Springer Verlag.

We wish to thank here our friend, Professor Czeslaw Bessaga, for reading a large part of the manuscript and making valuable suggestions and observations. We extend our thanks, for valuable remarks made in discussions and letters, to our colleagues and friends Professors W. J. Davis, D. van Dulst, T. Figiel, D. J. H. Garling, V. I. Gurariĭ, W. B. Johnson, M. I. Kadec, S. Kwapien, P. Masani, R. J. Nessel, K. I. Oskolkov and W. H. Ruckle. Furthermore, we thank our colleagues in the University of Amsterdam and Purdue University, who attended, in 1973/74, over a period of 5 months and 3 months respectively, our seminars on § 0 and on parts of §§ 8—11, for their stimulating interest and remarks. Also, we received useful comments from the participants to our lectures on parts of §§ 10, 11, at the Semester on approximation theory of the Stefan Banach International Mathematical Center in Warsaw, in 1975.

During the writing of this volume we have benefited by excellent working conditions at the Institute of Mathematics and at the National Institute for Scientific and Technical Creation in Bucharest, as well as at various Universities which we visited for periods of several months and we wish to express our gratitude to all who contributed to ensure these conditions. Our special thanks are due to Dr. Ing. Constantin Teodorescu, General Director of the National Institute for Scientific and Technical Creation and to Dr. Zoia Ceaușescu, Head of the Department of Mathematics at this Institute, without whose support this volume could not have been completed.

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After this book has been typeset, we added an Appendix, containing further notes and remarks and further bibliography. We extend our thanks to Professors B. V. Godun, M. I. Kadec, A. N. Plička and P. Terenzi, for correspondence concerning some parts of the Appendix. Finally, our thanks are due to Editura Academiei and to Springer Verlag, for undertaking the task of publishing this volume and for their help during its preparation.

May, 1980 IVAN SINGER

Contents

Chapter III

Generalizations of the Notion of a Basis

§ 0. Banach spaces which do not have the approximation property

We recall that a Banach space E is said to have the *approximation property* if the identity operator $I_E : E \to E$ can be approximated, uniformly on every compact subset of E, by continuous linear operators of finite rank (i.e., of finite-dimensional range), that is, if for every compact subset $Q \subset E$ and every $\varepsilon > 0$ there exists an endomorphism*) $v = v_{Q,\varepsilon} \in L(E, E)$ of finite rank, such that

$$\|v(x) - x\| < \varepsilon \qquad (x \in Q). \tag{0.1}$$

We have seen in Ch. I, § 17, theorem 17.3, that every Banach space E with a basis has the approximation property. The same argument also shows that a similar result holds**) for "basis" replaced by various generalizations of bases which will be introduced in the sequel (involving a sequence of finite rank operators on E converging pointwise to the identity operator I_E). Thus, the examples of separable Banach spaces failing to have the approximation property, which will be constructed in the present section, will give a (negative) solution to the basis problem (Ch. I, § 1, problem 1.1) and to the similar existence problems for these generalizations of bases.

The existence of separable Banach spaces without the approximation property has also other applications. For example, this fact implies, as we shall see below, that there exist Banach spaces with bases, whose conjugate space is separable and fails to have the approximation property. This latter result will be used in § 9 to show that there exists a separable Banach space E with the approximation property, having no basis (and which even fails to have any of the generalizations of bases mentioned above) and having E^* separable.

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*) We recall that $L(E, E)$ denotes the Banach space of all endomorphisms (continuous linear mappings of E into E), with the norm $\|u\| = \sup_{x \in E, \|x\| \leq 1} \|u(x)\|$.

**) However, we shall see in § 9, example 9.2, that the converse results are not true.

We shall give two different proofs of the existence of subspaces of c_0 and l^p $(2 < p < \infty)$ which fail to have the approximation property (theorems 0.1 and 0.2 below). Although these proofs have, essentially, the same underlying idea, each of them has its own interest. The first one will be of a constructive character, using a combinatorial lemma, while the second one will be a proof of existence, using a probabilistic lemma.

In both proofs we shall make use of the following sufficient condition for a Banach space E to fail the approximation property:

Lemma 0.1. a) *Let E be a Banach space. Assume that there exists a sequence* $\{\varphi_n\}_{n=0}^{\infty}$ *of linear functionals on* $L(E, E)$, *such that*

$$\varphi_n(I_E) = 1 \qquad (n = 3, 4, 5, \ldots), \tag{0.2}$$

$$\lim_{n\to\infty} \varphi_n(v) = 0 \qquad (v \in L(E, E),\ \dim v(E) < \infty), \tag{0.3}$$

a sequence $\{\mathscr{A}_n\}_{n=0}^{\infty}$ *of finite subsets of E and a sequence* $\alpha_n > 0$ $(n = 0, 1, 2, 3, \ldots)$ *with* $\sum_{n=0}^{\infty} \alpha_n < \infty$, *such that*

$$|\varphi_n(u) - \varphi_{n-1}(u)| \leqslant \alpha_n \max_{x\in\mathscr{A}_n} \|u(x)\| \qquad (u \in L(E, E),\ n = 0,1,2, \ldots), \tag{0.4}$$

$$\max_{x\in\mathscr{A}_n} \|x\| \to 0 \quad as \quad n \to \infty, \tag{0.5}$$

where $\varphi_{-1} = 0$. *Then E does not have the approximation property.*

b) *In particular, the last assumption* (0.4), (0.5) *is satisfied if* $\varphi_0 = \varphi_1 = \varphi_2 = 0$ *and if there exist decompositions*

$$\varphi_n = \frac{1}{2^n} \sum_{k=1}^{2^n} \varphi_{n,k}, \quad \varphi_{n-1} = \frac{1}{2^n} \sum_{k=1}^{2^n} \psi_{n,k} \qquad (n = 3,4,5, \ldots) \tag{0.6}$$

where $\varphi_{n,k}$ *and* $\psi_{n,k}$ $(k = 1, \ldots, 2^n;\ n = 3,4,5, \ldots)$ *are linear functionals on* $L(E, E)$, *and a family* $\{\mathscr{A}_{n,k}\}_{\substack{1\leqslant k\leqslant 2^n \\ 3\leqslant n<\infty}}$ *of finite subsets of E such that*

$$|\varphi_{n,k}(u) - \psi_{n,k}(u)| \leqslant \max_{x\in\mathscr{A}_{n,k}} \|u(x)\|$$

$$(u \in L(E, E),\ k = 1, \ldots, 2^n;\ n = 3,4, \ldots), \tag{0.7}$$

and such that the numbers $C_n = \max_{x\in \bigcup_{k=1}^{2^n} \mathscr{A}_{n,k}} \|x\|$ $(n = 3,4,5, \ldots)$ *satisfy*

$$\sum_{n=3}^{\infty} C_n < \infty. \tag{0.8}$$

Proof. a) Put

$$Q_0 = \{0\} \cup \bigcup_{n=0}^{\infty} \mathscr{A}_n. \tag{0.9}$$

Then, by (0.5), Q_0 is compact. Furthermore, by (0.4),

$$|\varphi_n(u) - \varphi_{n-1}(u)| \leqslant \alpha_n \sup_{x \in Q_0} \|u(x)\| \quad (u \in L(E, E), n = 0,1, \ldots), \tag{0.10}$$

whence, since $\alpha_n > 0$, $\sum_{n=0}^{\infty} \alpha_n < \infty$, the limits

$$\varphi(u) = \lim_{n\to\infty} \varphi_n(u) \qquad (u \in L(E, E)) \tag{0.11}$$

exist and satisfy the inequalities

$$|\varphi(u)| \leqslant \sum_{n=0}^{\infty} \alpha_n \sup_{x \in Q_0} \|u(x)\| \qquad (u \in L(E, E)). \tag{0.12}$$

Now let v be an arbitrary continuous linear operator of finite rank on E (i.e., $v \in L(E, E)$, $\dim v(E) < \infty$) and let $u = I_E - v$. Then, by (0.11) and (0.2), (0.3),

$$\varphi(u) = \lim_{n\to\infty} \varphi_n(u) = \lim_{n\to\infty} \varphi_n(I_E) - \lim_{n\to\infty} \varphi_n(v) = 1, \tag{0.13}$$

whence, by (0.12),

$$1 = |\varphi(u)| \leqslant \sum_{n=0}^{\infty} \alpha_n \sup_{x \in Q_0} \|x - v(x)\|,$$

so there is an $x_0 \in Q_0$ such that $\|x_0 - v(x_0)\| \geqslant \frac{1}{\sum_{n=0}^{\infty} \alpha_n}$. Thus, since $v \in L(E, E)$ was an arbitrary operator of finite rank, E does not have the approximation property.

b) By (0.8), we can choose*) a sequence $\{\alpha_n\}_{n=3}^{\infty}$ of positive numbers such that $\sum_{n=3}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \frac{C_n}{\alpha_n} = 0$. Put

$$\mathscr{A}_n = \bigcup_{k=1}^{2^n} \frac{1}{\alpha_n} \mathscr{A}_{n,k} \qquad (n = 3, 4, \ldots). \tag{0.14}$$

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*) Indeed, since $\sum_{j=3}^{\infty} C_j < \infty$, there exist integers $m_n \geqslant 2$ such that $\sum_{j=m_n+1}^{\infty} C_j < \frac{1}{n \cdot 2^{n+1}}$ $(n = 1, 2, \ldots)$. Hence, putting $\alpha_j = nC_j$ $(j = m_n + 1, \ldots$

Then $\mathscr{A}_n$ is finite and by (0.6), (0.7) we have

$$|\varphi_n(u) - \varphi_{n-1}(u)| = \left| \frac{1}{2^n} \sum_{k=1}^{2^n} (\varphi_{n,k}(u) - \psi_{n,k}(u)) \right| \leqslant \frac{1}{2^n} \sum_{k=1}^{2^n} \max_{x \in \mathscr{A}_{n,k}} \|u(x)\| \leqslant$$

$$\leqslant \max_{x \in \bigcup_{k=1}^{2^n} \mathscr{A}_{n,k}} \|u(x)\| = \alpha_n \max_{x \in \mathscr{A}_n} \|u(x)\| \quad (u \in L(E, E), n = 3, 4, \ldots),$$

which, together with $\varphi_0 = \varphi_1 = \varphi_2 = 0$, gives (0.4) (with any finite sets $\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2$). Also,

$$\max_{x \in \mathscr{A}_n} \|x\| = \max_{x \in \bigcup_{k=1}^{2^n} \frac{1}{\alpha_n} \mathscr{A}_{n,k}} \|x\| \leqslant \frac{C_n}{\alpha_n} \to 0 \text{ as } n \to \infty,$$

and thus we have (0.5), which completes the proof.

Remark 0.1. a) One can also give the following geometric interpretation of the above proof of part a): Since each φ_n is linear on $L(E, E)$, so is φ defined by (0.11) (we have used this above, in (0.13)). Moreover, by (0.12) φ is also continuous for the topology of uniform convergence on compact subsets of E (since a neighborhood base of 0 for that topology is*) the family $V_{Q,r}(0) = \{u \in L(E, E) | \sup_{u \in Q} \|u(x)\| < r\}$, where $Q \subset E$ is compact and $r > 0$, and since for each $\varepsilon > 0$ there exists $V_{Q_\varepsilon, r_\varepsilon}(0)$, namely, $Q_\varepsilon = Q_0$ and $r_\varepsilon = \frac{\varepsilon}{\sum_{n=0}^{\infty} \alpha_n}$, such that $|\varphi(u)| < \varepsilon$ for all $u \in V_{Q_\varepsilon, r_\varepsilon}(0)$). Since by (0.2) and (0.3), $\varphi(I_E) = \lim_{n\to\infty} \varphi_n(I_E) = 1$ and $\varphi(v) = \lim_{n\to\infty} \varphi_n(v) = 0$ for each $v \in L(E, E)$ with $\dim v(E) < \infty$, φ separates I_E from the closure of the linear subspace $\{v \in L(E, E) | \dim v(E) < \infty\}$ of $L(E, E)$ in the topology of compact convergence, which is equivalent to the fact that E does not have the approximation property.

$\ldots, m_{n+1};\ n = 1, 2, \ldots)$, we have $\frac{C_j}{\alpha_j} = \frac{1}{n}$ $(j = m_n + 1, \ldots, m_{n+1})$, which $\to 0$ as $j \to \infty$, and $\sum_{j=m_1+1}^{\infty} \alpha_j = \sum_{n=1}^{\infty} n \sum_{j=m_n+1}^{m_{n+1}} C_j \leqslant \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$, so it remains to take arbitrary $\alpha_3, \ldots, \alpha_{m_1} > 0$.

*) See e.g. [355], p. 79; this topology is also called the "topology of compact convergence". Condition (0.4) means that, for each n, φ_n is so "near" to φ_{n-1}, as to guarantee the existence and the continuity of φ for this topology.

b) Let us observe that any decompositions $\varphi_n = \frac{1}{2^n} \sum_{k=1}^{2^n} \varphi_{n,k}$ $(n = 3, 4, 5, \ldots)$ imply the obvious decompositions $\varphi_{n-1} = \frac{1}{2^n} \sum_{k=1}^{2^n} \psi'_{n,k}$ $(n = 3, 4, 5, \ldots; \varphi_2 = 0)$, where

$$\psi'_{n,k} = \begin{cases} 2\varphi_{n-1,k} & \text{for } k = 1, \ldots, 2^{n-1} \\ 0 & \text{for } k = 2^n; \end{cases}$$

however, we shall work with other decompositions (0.6).

Let us explain now how we shall define the subspaces E of c_0 and l^p $(2 < p < \infty)$, the linear functionals φ_n on $L(E, E)$ and the finite subsets $\mathscr{A}_n$ of E satisfying the conditions of lemma 0.1. We recall that if E is a Banach space and if $(x_i, f_i)_{i \in \mathscr{M}}$ $(\{x_i\}_{i \in \mathscr{M}} \subset E, \{f_i\}_{i \in \mathscr{M}} \subset E^*)$ is a countable biorthogonal system, then for any finite non-empty subset M of the index set $\mathscr{M}$, the *average trace* over M is the linear functional on $L(E, E)$ defined by

$$\mathscr{T}(M; u) = \frac{1}{|M|} \sum_{i \in M} f_i(u(x_i)) \qquad (u \in L(E, E)), \tag{0.15}$$

where $|M|$ denotes*) the cardinality of the set M.

The desired subspaces E of c_0 and l^p $(2 \leq p < \infty)$ will be of the form $E = [x_i]_{i \in \mathscr{M}}$, where $(x_i, f_i)_{i \in \mathscr{M}}$ $(\{x_i\}_{i \in \mathscr{M}} \subset E, \{f_i\}_{i \in \mathscr{M}} \subset E^*)$ is a suitable countable E-complete biorthogonal system (the index set $\mathscr{M}$ will be chosen later). Each functional $\varphi_n(u)$ (except $\varphi_0 = \varphi_1 = \varphi_2 = 0$ in the first proof) will be of the form $\mathscr{T}(M_n; u)$, for a suitable sequence $\{M_n\}$ of pairwise disjoint finite subsets**) of $\mathscr{M}$ such that $\cup M_n = \mathscr{M}$ (thus, the nearness condition (0.4) will mean that the "jump" of $\mathscr{T}$ from M_{n-1} to M_n is***) small enough) and each $\mathscr{A}_n$ will be a finite set of finite linear combinations of the x_i's.

Let us first observe that by (0.15) and biorthogonality we have, for any finite non-empty subset $M \subset \mathscr{M}$,

$$\mathscr{T}(M; I_E) = \frac{1}{|M|} \sum_{i \in M} f_i(x_i) = 1,$$

and hence the linear functionals $\varphi_n(u) = \mathscr{T}(M_n; u)$ will certainly satisfy (0.2). Furthermore, if $v \in L(E, E)$ is an operator of rank 1 of the form

$$v(x) = f_0(x) x_{i_0} \qquad (x \in E), \tag{0.16}$$

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*) In the other sections we shall use the notation card M.

**) In theorems 0.1 and 0.2 we shall have $|M_n| = 2^{2n-1}$ and $|M_n| = 2^n$ $(n = 1, 2, \ldots)$, respectively.

***) For a consequence of this fact see remark 0.5.

where $f_0 \in E^*$, $i_0 \in I$, then, by biorthogonality,

$$\varphi_n(v) = \mathscr{T}(M_n; v) = \frac{1}{|M_n|} \sum_{i \in M_n} f_i(f_0(x_i)x_{i_0}) =$$

$$= \frac{1}{|M_n|} \sum_{i \in M_n} f_0(x_i)\delta_{i,i_0} \qquad (n = 0,1,2, \ldots).$$

However, when n is large enough, we have $i_0 \notin M_n$, so $\delta_{i,i_0} = 0$ $(i \in M_n)$, and hence $\varphi_n(v) = 0$. Consequently, since the set of all operators of rank 1 of the form (0.16) is complete, for the norm topology of $L(E, E)$, in the set of all finite rank operators on E, in order to prove (0.3) it is enough to show that $\sup_{0 \leqslant n < \infty} \|\varphi_n\| < \infty$. But, by (0.10) and $\alpha_j > 0$, $\sum_{j=0}^{\infty} \alpha_j = 1$, we have

$$|\varphi_n(u)| = \left| \sum_{j=0}^{n} (\varphi_j(u) - \varphi_{j-1}(u)) \right| \leqslant \sum_{j=0}^{n} \alpha_j \sup_{x \in Q_0} \|u(x)\| \leqslant$$

$$\leqslant \left(\sum_{j=0}^{\infty} \alpha_j \sup_{x \in Q_0} \|x\| \right) \|u\| \qquad (u \in L(E, E),\ n = 0, 1, 2, \ldots),$$

whence $\sup_{0 \leqslant n < \infty} \|\varphi_n\| < \infty$,*) which will prove (0.3), provided that we shall have (0.4) and (0.5) assured. Thus, it will be sufficient to concentrate our efforts to achieve (0.4) and (0.5).

In both proofs below it will be convenient to replace c_0 and l^p $(2 < p < \infty)$, by the (isometric) spaces $c_0(\Gamma)$ and $l^p(\Gamma)$ $(2 < p < \infty)$ for a suitable countable set Γ, so the elements of the desired space E will be**) scalar-valued functions on Γ. We shall take the x_i's to be certain functions on Γ with finite supports supp x_i and such that

$$|x_i(g)| = 1 \qquad (g \in \operatorname{supp} x_i;\ i \in \mathscr{M}), \tag{0.17}$$

whence $x_i \in c_0(\Gamma) \subset l^p(\Gamma)$, $\|x_i\|_{c_0(\Gamma)} = 1$, $\|x_i\|_{l^p(\Gamma)} = |\operatorname{supp} x_i|^{\frac{1}{p}}$ $(1 \leqslant p < \infty)$, and then we shall put $E = [x_i]_{i \in \mathscr{M}}$ in $c_0(\Gamma)$, respectively

*) Actually, we shall see in a moment that the biorthogonal system $(x_i, f_i)_{i \in \mathscr{M}}$ will be chosen so as to satisfy $\|x_i\| \|f_i\| = 1$ $(i \in \mathscr{M})$, whence $|\varphi_n(u)| = |\mathscr{T}(M_n, u)| \leqslant \|u\|$ and thus, by (0.2), $\|\varphi_n\| = 1$ for all n.

**) For any countable set $\Gamma = \{\gamma_n\}$, $c_0(\Gamma)$ is the Banach space of all functions $x(.)$ on Γ such that $\{x(\gamma_n)\}_{n=1}^{\infty} \in c_0$, with $\|x(.)\|_{c_0(\Gamma)} = \|\{x(\gamma_n)\}\|_{c_0} = \sup_{1 \leqslant n < \infty} |x(\gamma_n)|$; the case of $l^p(\Gamma)$ is similar.

in $l^p(\Gamma)$. Naturally, the supports of the x_i's cannot be pairwise disjoint (since if they were, then $\{x_i\}_{i\in\mathcal{M}}$ would be a bloc basic sequence with respect to the unit vector basis of $c_0(\Gamma)$ and $l^p(\Gamma)$, and hence a basis of $E=[x_i]_{i\in\mathcal{M}}$), but we shall choose the x_i's in such a way that the sequence $\{x_i\}_{i\in\mathcal{M}}$ will be orthogonal in $l^2(\Gamma)$. Therefore, defining

$$f_i(x)=\frac{1}{\|x_i\|^2_{l^2(\Gamma)}}(x, x_i)=\frac{1}{|\operatorname{supp} x_i|}\sum_{g\in\operatorname{supp} x_i} x(g)\overline{x_i(g)} \quad (x\in E,\ i\in\mathcal{M}),$$

we shall have $f_i\in E^*$ and $(x_i, f_i)_{i\in\mathcal{M}}$ will be an E-complete biorthogonal system; also, clearly,

$$\|f_i\|=\frac{1}{|\operatorname{supp} x_i|}\|x_i\|_{l^q(\Gamma)}=|\operatorname{supp} x_i|^{\frac{1}{q}-1},$$

where $\frac{1}{p}+\frac{1}{q}=1 \quad (1\leqslant p\leqslant\infty;\ \frac{1}{\infty}=0)$, whence $\|x_i\|\,\|f_i\|=1$ $(i\in\mathcal{M})$. Then, by the above choice of the functionals φ_n,

$$\varphi_n(u)-\varphi_{n-1}(u)=\frac{1}{|M_n|}\sum_{i\in M_n} D_i\sum_{g\in\Gamma} u(x_i)(g)\overline{x_i(g)}-$$

$$-\frac{1}{|M_{n-1}|}\sum_{i\in M_{n-1}} D_i\sum_{g\in\Gamma} u(x_i)(g)\overline{x_i(g)}=$$

$$=\sum_{g\in\Gamma_n} u\left(\frac{1}{|M_n|}\sum_{i\in M_n} D_i\overline{x_i(g)}x_i-\frac{1}{|M_{n-1}|}\sum_{i\in M_{n-1}} D_i\overline{x_i(g)}x_i\right)(g)$$

$$(u\in L(E, E),\ n=0, 1, \ldots),$$

where $\Gamma_n=\bigcup_{i\in M_n\cup M_{n-1}}\operatorname{supp} x_i$ and $D_i=\frac{1}{|\operatorname{supp} x_i|}$. This formula can be further simplified by choosing the x_i's so that $D_i=\frac{1}{|\operatorname{supp} x_i|}$ is constant on each M_n, say $D_i=\delta_n$ $(i\in M_n)$. Then, putting

$$y_g^n=\frac{\delta_n}{|M_n|}\sum_{i\in M_n}\overline{x_i(g)}x_i-\frac{\delta_{n-1}}{|M_{n-1}|}\sum_{i\in M_{n-1}}\overline{x_i(g)}x_i \quad (g\in\Gamma_n), \qquad (0.18)$$

we obtain

$$|\varphi_n(u) - \varphi_{n-1}(u)| = |\sum_{g \in \Gamma_n} u(y_g^n)(g)| \leqslant \sum_{g \in \Gamma_n} |u(y_g^n)(g)| \leqslant$$

$$\leqslant \sum_{g \in \Gamma_n} \|u(y_g^n)\|_{l^p(\Gamma)} \leqslant |\Gamma_n| \max_{g \in \Gamma_n} \|u(y_g^n)\|_{l^p(\Gamma)} \quad (u \in L(E, E),\ n = 0,1, \ldots),$$

where $1 \leqslant p \leqslant \infty$. Thus, it is intuitively clear that if we could take the elements of the finite sets $\mathscr{A}_n$ to be of a suitable similar form*) to the $\frac{|\Gamma_n|}{\alpha_n} y_g^n$'s, then (0.4) would be satisfied and we would have to concentrate only to obtain such elements with norms small enough to assure (0.5) as well. Actually, in order to define such elements, in the first proof we shall also exploit that each φ_n admits two different decompositions, as in (0.6) (the functionals $\varphi_{n,k}(u)$, $\psi_{n,k}(u)$ will be average traces $\mathscr{T}(M_{n,k}; u)$, $\mathscr{T}(L_{n,k}; u)$ over some suitable subsets $M_{n,k}$, $L_{n,k}$ of M_n and M_{n-1}, respectively), while in the second proof we shall be able to write each f_i in two different forms and to exploit both of them. This will lead to (0.4), (0.5) for $\|\cdot\|_{l^p(\Gamma)}$, where $2 < p < \infty$.

To this end, we shall take Γ to be a countable union of certain pairwise disjoint finite groups G_n and we shall take each x_i to be a certain linear combination, with coefficients ± 1, of a constant finite number**) of characters (or, equivalently, Walsh functions) on different groups G_n, extended by 0 on $\Gamma \setminus G_n$. Then, in order to prove the existence of elements of similar form to (0.18), with norms growing not too fast, we shall need (in each proof) a lemma on the existence of a "good" partition of the set of all characters of the groups G_n entering in the definition of Γ. Also, in the first proof we shall use a combinatorial lemma which will enable us to "stretch out" Γ so much as to obtain disjoint supports for the five summands in the definition of the functions x_i and supports with small intersection for the summands in the elements of the sets $\mathscr{A}_n$ (thus, this lemma will help to make our choice of Γ and of the x_i's and $\mathscr{A}_n$'s), while in the second proof we shall use, instead, a probabilistic lemma, to show that "good" partitions of sets of characters and "good" choices of the above mentioned coefficients ± 1 in the definition of the x_i's exist.

Let us pass now to the first proof. We start with the lemma on the partition of the set of characters, mentioned above. For convenience,

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*) In fact, in both proofs we shall take the elements of $\mathscr{A}_n$ to be linear combinations of the x_i's with $i \in M_n \cup M_{n-1}$.

**) In the first proof this number will be five, while in the second proof it will be two.

in the sequel we shall use the notation $\|.\|_p$ instead of $\|.\|_{l^p(G)}$ (where G is any set).

Lemma 0.2. *Let n be a positive integer, let G be the (abelian) group $\{-1,1\}^n$ (of all functions from $\{1, 2, \ldots, n\}$ into $\{-1, 1\}$, with pointwise multiplication) and let $H \subset C(G)$ be the set of all characters*) of G. Then there exist two disjoint subsets W^+, W^- of H with cardinalities satisfying*

$$|W^+| = |W^-| = \frac{1}{2}|G| = 2^{n-1} \qquad (0.19)$$

(hence $W^+ \cup W^- = H$), such that

$$\|\sum_{w\in W^+} w - \sum_{w\in W^-} w\|_\infty \leqslant 2^{1+\left[\frac{n}{2}\right]}. \qquad (0.20)$$

Proof. Let $k = \left[\frac{n-1}{2}\right]$ and let $s = n - 2k$, hence $s \geqslant 1$ and $n = 2k + s$. Define $r_j \in H$ by

$$r_j(g) = g(j) \qquad (g \in G,\ j = 1, \ldots, n) \qquad (0.21)$$

and $y_0 \in C(G)$ by

$$y_0 = \prod_{j=1}^{k} (1 - r_{2j-1} - r_{2j} - r_{2j-1}r_{2j}) \prod_{t=1}^{s} (1 - r_{2k+t}). \qquad (0.22)$$

Now, by (0.21), each product $\prod_{j\in S} r_j$, where $S \subset \{1, 2, \ldots, n\}$, is a character $w = w_S \in H$ and the set of all such products coincides with the set H of all characters (since it contains 2^n distinct elements, so it has the same cardinality as H). Since at "term by term multiplication" in (0.22) each such product occurs exactly once, we obtain $y_0 = \sum_{w\in H} c_w w$, where each coefficient c_w is either 1 or -1. Put

$$W^+ = \{w \in H | c_w = 1\},\quad W^- = \{w \in H | c_w = -1\}, \qquad (0.23)$$

*) I.e., of all homomorphisms of G into $\{\zeta \,|\, |\zeta| = 1\}$. Since $G = \{-1, 1\}^n$, all characters have real values, i.e., ± 1.

and let e denote the unit of G. Since $r_{2k+s}(e) = r_n(e) = e(n) = 1$, the last factor in (0.22) is 0 at e, so $y_0(e) = 0$, whence, since $w(e) = 1$ ($w \in H$), we obtain

$$|W^+| - |W^-| = \sum_{w \in H} c_w = \sum_{w \in H} c_w w(e) = y_0(e) = 0,$$

which proves (0.19). Finally, for any $g \in G$, by (0.22) $y_0(g)$ is a product of $k + s = 1 + \left[\frac{n}{2}\right]$ numbers, each of them being 2, -2 or 0, whence, since $y_0 = \sum_{w \in H} c_w w = \sum_{w \in W^+} w - \sum_{w \in W^-} w$, we obtain (0.20), which completes the proof of lemma 0.2.

Remark 0.2. Let $2^{[1,n]}$ denote the (abelian) group of all subsets of the set $[1, n] = \{1, 2, \ldots, n\}$, with the group operation $S_1 S_2 = S_1 \dotdiv S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$ (the symmetric difference of the subsets S_1, S_2 of $[1, n]$). Then $G = \{-1, 1\}^n$ is isomorphic to $2^{[1,n]}$, by the mapping $g \to S_g = \{j \in [1, n] \mid g(j) = -1\}$, and $C(G)$ is isomorphic to $l^\infty_{2^n}$ (the Banach space of all scalar functions $\xi = \{\xi(S)\}_{S \in 2^{[1,n]}}$ with the norm $\|\xi\| = \max_{S \in 2^{[1,n]}} |\xi(s)|$), by the mapping $x \to \xi_x$, where $\xi_x(S_g) = x(g)$ ($g \in G$). This latter isomorphism carries H onto the set of all Walsh functions $\beta_S(S_g) = (-1)^{|S \cap S_g|}$ and $\{r_j\}_{j=1}^n$ onto the set of all Rademacher functions $\rho_j(S_g) = 1$ for $S_g \not\ni j$ and -1 for $S_g \ni j$ ($j = 1, \ldots, n$). This motivates the notations w and r_j used above.

Let us give now the combinatorial lemma mentioned above:

Lemma 0.3. *Let $n \geq 4$ be a positive integer and let $(A_i)_{i \in I}$, $(B_j)_{j \in J}$ be two families of mutually disjoint sets with cardinalities satisfying*

$$|A_i| = \frac{1}{2} |I| = 2^{n-2} \qquad (i \in I), \tag{0.24}$$

$$|B_j| = \frac{1}{2} |J| = 2^{n-1} \qquad (j \in J) \tag{0.25}$$

(i.e., there are 2^{n-1} sets A_i, each of cardinality 2^{n-2} and there are 2^n sets B_j, each of cardinality 2^{n-1}). Then there exists a function $p: \bigcup_{j \in J} B_j \to \bigcup_{i \in I} A_i$ such that

$$|p(B_j) \cap A_i| = 1 \qquad (i \in I,\ j \in J), \tag{0.26}$$

$$|p^{-1}(a)| = 4 \qquad (a \in \bigcup_{i \in I} A_i), \tag{0.27}$$

$$|p(B_j) \cap p(B_{j'})| \leq 2 \qquad (j, j' \in J,\ j \neq j') \tag{0.28}$$