Lecture Notes in Mathematics

Rolf Gohm

Noncommutative Stationary Processes

1839

$$\begin{array}{ccc} \mathcal{A}, \phi_{\mathcal{A}} & \xrightarrow{\quad S \quad} & \mathcal{B}, \phi_{\mathcal{B}} \\ \cap & & \cap \\ \mathcal{B}(\mathcal{G}), \Omega_{\mathcal{G}} & \xrightarrow{\ Z\ } & \mathcal{B}(\mathcal{H}), \Omega_{\mathcal{H}} \end{array}$$

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Preface

Research on noncommutative stationary processes leads to an interesting interplay between operator algebraic and probabilistic topics. Thus it is always an invitation to an exchange of ideas between different fields. We explore some new paths into this territory in this book. The presentation proceeds rather systematically and elaborates many connections to already known results as well as some applications. It should be accessible to anyone who has mastered the basics of operator algebras and noncommutative probability but, concentrating on new material, it is no substitute for the study of the older sources (mentioned in the text at appropriate places). For a quick orientation see the Summary on the following page and the Introduction. There are also additional introductions in the beginning of each chapter.

The text is a revised version of a manuscript entitled 'Elements of a spatial theory for noncommutative stationary processes with discrete time index', which has been written by the author as a habilitation thesis (Greifswald, 2002). It is impossible to give a complete picture of all the mathematical influences on me which shaped this work. I want to thank all who have been engaged in discussions with me. Additionally I want to point out that B. Kümmerer and his students C. Hertfelder and T. Lang, sharing some of their conceptions with me in an early stage, influenced the conception of this work. Getting involved with the research of C. Köstler, B.V.R. Bhat, U. Franz and M. Schürmann broadened my thinking about noncommutative probability. Special thanks to M. Schürmann for always supporting me in my struggle to find enough time to write. Thanks also to B. Kümmerer and to the referees of the original manuscript for many useful remarks and suggestions leading to improvements in the final version. The financial support by the DFG is also gratefully acknowledged.

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Summary

In the first chapter we consider normal unital completely positive maps on von Neumann algebras respecting normal states and study the problem to find normal unital completely positive extensions acting on all bounded operators of the GNS-Hilbert spaces and respecting the corresponding cyclic vectors. We show that there exists a duality relating this problem to a dilation problem on the commutants. Some explicit examples are given.

In the second chapter we review different notions of noncommutative Markov processes, emphasizing the structure of a coupling representation. We derive related results on Cuntz algebra representations and on endomorphisms. In particular we prove a conjugacy result which turns out to be closely related to Kümmerer-Maassen-scattering theory. The extension theory of the first chapter applied to the transition operators of the Markov processes can be used in a new criterion for asymptotic completeness. We also give an interpretation in terms of entangled states.

In the third chapter we give an axiomatic approach to time evolutions of stationary processes which are non-Markovian in general but adapted to a given filtration. We call this an adapted endomorphism. In many cases it can be written as an infinite product of automorphisms which are localized with respect to the filtration. Again considering representations on GNS-Hilbert spaces we define adapted isometries and undertake a detailed study of them in the situation where the filtration can be factorized as a tensor product. Then it turns out that the same ergodic properties which have been used in the second chapter to determine asymptotic completeness now determine the asymptotics of nonlinear prediction errors for the implemented process and solve the problem of unitarity of an adapted isometry.

In the fourth chapter we give examples. In particular we show how commutative processes fit into the scheme and that by choosing suitable noncommutative filtrations and adapted endomorphisms our criteria give an answer to a question about subfactors in the theory of von Neumann algebras, namely when the range of the endomorphism is a proper subfactor.

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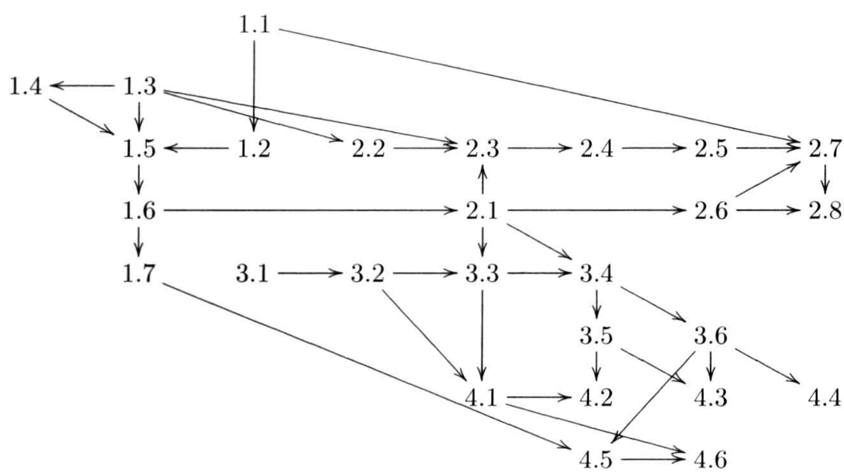
Flow Diagram for the Sections



Introduction

This work belongs to a field called quantum probability or noncommutative probability. The first name emphasizes the origins in quantum theory and the attempts to achieve a conceptual understanding of the new probabilistic features of this theory as well as the applications to physics which such a clarification can offer in return. The second name, which should be read as not necessarily commutative probability, puts the subject into the broader program of noncommutative mathematics and emphasizes the development of mathematical structures. The field has grown large and we do not intend to give a survey here but refer to the books [Da76, Me91, Pa92, Bi95, Ho01, QPC03] for different ways of approaching it. Probability theory in the usual sense appears as a part which is referred to as classical or commutative.

The core of classical probability consists of the theory of stochastic processes and in this respect noncommutative probability follows its predecessor. But the additional freedom to use noncommutative algebras offers vast new possibilities. From the beginning in quantum theory it has been realized that in particular operator algebras offer a rich source, i.e. algebras of operators on a Hilbert space. Especially since the eighties of the last century it has been shown that on a Hilbert space with a special structure, the Fock space, many aspects of classical probability and even rather advanced ones, can be reconstructed in the noncommutative framework in a revealing way. One of the highlights is a theory of noncommutative stochastic integration by R.L. Hudson and K.R. Parthasarathy which can be used as a tool to realize many noncommutative stochastic processes. Also the fundamental processes of classical probability, such as Brownian motion, appear again and they are now parts of noncommutative structures and processes in a very interesting way.

Other aspects come into play if one tries to use the theory of operator algebras more explicitly. This is also done in this work. An important starting point for us is the work done by B. Kümmerer since the eighties of the last century. Here the main idea has been to consider stationary Markov processes. In classical probability Markov processes received by far the most attention due to the richness of their algebraic and analytic properties. Stationarity, i.e. the dependence of probability distributions only on time differences, yields connections to further fields of mathematics such as dynamical systems and ergodic theory. The same is true in noncommutative probability. The structure theory of noncommutative stationary Markov processes generalizes many classical properties and exhibits new ones, giving also insights which relate probabilistic notions and models in quantum physics. Stationarity gives rise to time evolutions which are endomorphisms of operator algebras and thus provides a link between research in noncommutative probability and in operator algebras. In this theory the role of the Hilbert space becomes secondary and the abstract structure theory of operator algebras, especially von Neumann algebras, comes into view.

Here we have arrived at a very interesting feature of the theory of operator algebras. While they may be defined as algebras of operators on a Hilbert space, the most interesting of them, such as C^*-algebras or von Neumann algebras, also have intrinsic characterizations. Thus their theory can be developed intrinsically, what we have called abstract structure theory above, or one can study representation theory, also called spatial theory, which uses representations of the elements of the algebra as operators on a Hilbert space. Of course, many properties are best understood by cleverly combining both approaches.

Combining both approaches should also be useful in considering noncommutative stochastic processes. A general idea behind this work can be formulated as follows: For stationary Markov processes or stationary processes in general which can be defined in an abstract way, study some of their properties which become more accessible by including the spatial point of view.

Similar endeavours are of course implicit in many works on noncommutative probability, but starting from abstract stationary processes we can do it more explicitly. The text is based on the author's habilitation thesis with the more lengthy and more precise title 'Elements of a spatial theory for noncommutative stationary processes with discrete time index'. We have already explained what we mean by 'spatial'. The precise titel also makes clear that we do not intend to write a survey about all that is known about noncommutative processes. In particular the restriction to discrete time steps puts aside a lot of work done by quantum probabilists. While there are parts of this text where generalization to continuous time is rather obvious there are other parts where it is not, and it seems better to think about such things at a separate place.

On the other hand, by this restriction we open up the possibility to discard many technicalities, to concentrate on very basic problems and to discuss the issue how a systematic theory of noncommutative stationary processes may look like. Guided by the operator algebraic and in particular the corresponding spatial point of view we explore features which we think should be elements of a general theory. We will see analogies to the theory of commutative stationary processes and phenomena which only occur in the noncommutative setting.

It is encouraging that on our way we also achieve a better understanding of the already known approaches and that some applications to physics show up. It is clear however that many things remain to be done. The subject is still not mature enough for a definite top-down axiomatic treatment and there is much room for mental experimentation.

Now let us become more specific. Classical Markov processes are determined by their transition operators and are often identified with them, while for the noncommutative Markov processes mentioned above this is no longer the case. A very natural link between the classical and the noncommutative case occurs when they are both present together, related by extension respectively by restriction. Using spatial theory, more precisely the GNS-construction, we introduce the notion of an extended transition operator which acts on all bounded operators on the GNS-Hilbert space. This notion plays a central role in our theory and many sections study the delicate ways how extended transition encodes probabilistic information. While the original transition operator may act on a commutative or noncommutative algebra, the extended transition operator always acts on a noncommutative algebra and thus can only be considered as a probabilistic object if one includes noncommutative probability theory. In Chapter 1 we give the definitions and explore directly the relations between transition and extended transition. There exists a kind of duality with a dilation problem arising from the duality between algebras and commutants, and studying these problems together sheds some light on both. We introduce the concept of a weak tensor dilation in order to formulate a one-to-one correspondence between certain extensions and dilations. The study of this duality is the unifying theme of Chapter 1. We also give some examples where the extensions can be explicitly computed.

In Chapter 2 we study the significance of extended transition for Markov processes. In B. Kümmerer's theory of noncommutative stationary Markov processes their coupling structure is emphasized. Such a coupling representation may be seen as a mathematical structure theorem about noncommutative Markov processes or as a physical model describing the composition of a quantum system as a small open system acted upon by a large reservoir governed by noise. In this context we now recognize that the usefulness of extended transition lies mainly in the fact that it encodes information on the coupling which is not contained in the original transition operator of the Markov process. This encoding of the relevant information into a new kind of transition operator puts the line of thought nearer to what is usual in classical probability. This becomes even more transparent if one takes spatial theory one step further and extends the whole Markov process to an extended Markov process acting on all bounded operators on the corresponding GNS-Hilbert space. Here we notice a connection to the theory of weak Markov processes initiated by B.V.R. Bhat and K.R. Parthasarathy and elaborated by Bhat during the nineties of the last century. To connect Kümmerer's and Bhat's approaches by an extension procedure seems to be a natural idea which has not been studied up to now, and we describe how it can be done in our context.

For a future similar treatment of processes in continuous time this also indicates a link to the stochastic calculus on Fock space mentioned earlier. In fact, the invariant state of our extended process is a vector state, as is the Fock vacuum which is in most cases the state chosen to represent processes on Fock space. The possibility to get pure states by extension is one of the most interesting features of noncommutativity. Part of the interest in Fock space calculus always has been the embedding of various processes, such as Brownian motion, Poisson processes, Lévy processes, Markov processes etc., commutative as well as noncommutative, into the operators on Fock space. Certainly here are some natural possibilities for investigations in the future.

In Chapter 2 we also explore the features which the endomorphisms arising as time evolutions of the processes inherit from the coupling representation. This results in particular in what may be called coupling representations of Cuntz algebras. A common background is provided by the theory of dilations of completely positive maps by endomorphisms, and in this rerspect we see many discrete analogues of concepts arising in W. Arveson's theory of E_0-semigroups.

The study of cocycles and coboundaries connecting the full time evolution to the evolution of the reservoir leads to an application of our theory to Kümmerer-Maassen-scattering theory. In particular we show how this scattering theory for Markov processes can be seen in the light of a conjugacy problem on the extended level which seems to be somewhat simpler than the original one and which yields a new criterion for asymptotic completeness. An interpretation involving entanglement of states also becomes transparent by the extension picture. Quantum information theory has recently rediscovered the significance of the study of entanglement and of related quantities. Here we have a surprising connection with noncommutative probability theory. Some interesting possibilities for computations in concrete physical models also arise at this point.

Starting with Chapter 3 we propose a way to study stationary processes without a Markov property. We have already mentioned that stationarity yields a rich mathematical structure and deserves a study on its own. Further, an important connection to the theory of endomorphisms of operator algebras rests on stationarity and one can thus try to go beyond Markovianity in this respect. We avoid becoming too broad and unspecific by postulating adaptedness to a filtration generated by independent variables, and independence here means tensor-independence. This leads to the concept of an adapted endomorphism. There are various ways to motivate this concept. First, in the theory of positive definite sequences and their isometric dilations on a Hilbert space it has already been studied, in different terminology. Second, it is a natural generalization of the coupling representation for Markov processes mentioned above. Third, category theory encourages us to express all our notions by suitable morphisms and this should also be done for the notion of adaptedness. We study all these motivations in the beginning of Chapter 3 and then turn to applications for stationary processes.

It turns out that in many cases an adapted endomorphism can be written as an infinite product of automorphisms. The factors of this product give some information which is localized with respect to the filtration and can be thought of as building the endomorphism step by step. Such a successive adding of time steps of the process may be seen as a kind of 'horizontal' extension procedure, not to be confused with the 'vertical' extensions considered earlier which enlarge the algebras in order to encode better the information about a fixed time step. But both procedures can be combined. In fact, again it turns out that it is the spatial theory which makes some features more easily accessible.

The applications to stationary processes take, in a first run, the form of a structure theory for adapted isometries on tensor products of Hilbert spaces. Taking a hint from transition operators and extended transition operators of Markov processes we again define certain completely positive maps which encode properties in an efficient way. We even get certain dualities between Markov processes and non-Markovian processes with this point of view. These dualities rely on the fact that the same ergodic properties of completely positive maps which are essential for our treatment of asymptotic completeness in Kümmerer-Maassen scattering theory also determine the asymptotics of nonlinear prediction errors and answer the question whether an adapted endomorphism is an automorphism or not.

While such product representations for endomorphisms have occurred occasionally in the literature, even in the work of prominent operator algebraists such as A. Connes and V.F.R. Jones and in quantum field theory in the form developed by R. Longo, there exists, to the knowledge of the author, no attempt for a general theory of product representations as such. Certainly such a theory will be difficult, but in a way these difficulties cannot be avoided if one wants to go beyond Markovianity. The work done here can only be tentative in this respect, giving hints how our spatial concepts may be useful in such a program.

Probably one has to study special cases to find the most promising directions of future research. Chapter 4 provides a modest start and treats the rather abstract framework of Chapter 3 for concrete examples. This is more than an illustration of the previous results because in all cases there are specific questions natural for a certain class of examples, and comparing different such classes then leads to interesting new problems. First we cast commutative stationary adapted processes into the language of adapted endomorphisms, which is a rather uncommon point of view in classical probability. More elaboration of the spatial theory remains to be done here, but we show how the computation of nonlinear prediction errors works in this case. Noncommutative examples include Clifford algebras and their generalizations which have some features simplifying the computations. Perhaps the most interesting but also rather difficult case concerns filtrations given by tensor products of matrices. Our criteria can be used to determine whether the range of an adapted endomorphism is a proper subfactor of the hyperfinite II_1−factor, making contact to a field of research in operator algebras. However here we have included only the most immediate observations, and studying these connections is certainly a work on its own. We close this work with some surprising observations about extensions of adapted endomorphisms, exhibiting phenomena which cannot occur for Markov processes. Remarkable in this respect is the role of matrices which in quantum information theory represent certain control gates.

There is also an Appendix containing results about unital completely positive maps which occur in many places of the main text. These maps are the transition operators for noncommutative processes, and on the technical level it is the structure theory of these maps which underlies many of our results. It is therefore recommended to take an early look at the Appendix.

It should be clear by these comments that a lot of further work can be done on these topics, and it is the author's hope that the presentation in this book provides a helpful starting point for further attempts in such directions.

Preliminaries and notation

$\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$

Hilbert spaces are assumed to be complex and separable: $\mathcal{G},\ \mathcal{H},\ \mathcal{K},\ \mathcal{P}, \ldots$

The scalar product is antilinear in the first and linear in the second component.

Often $\xi \in \mathcal{G},\ \xi \in \mathcal{H},\ \eta \in \mathcal{K},\ \eta \in \mathcal{P}$.

Ω is a unit vector, often arising from a GNS-construction.

Isometries, unitaries: $v,\ u$

Projection on a Hilbert space always means orthogonal projection: $p,\ q$

p_ξ denotes the one-dimensional projection onto $\mathbb{C}\xi$. Sometimes we also use Dirac notation, for example $p_\xi = |\,\xi\rangle\langle\xi\,|$.

M_n denotes the $n \times n$-matrices with complex entries,

$\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$.

'stop' means: strong operator topology

'wop' means: weak operator topology

$\mathcal{T}(\mathcal{H})$ trace class operators on $\mathcal{H}$

$\mathcal{T}_+^1(\mathcal{H})$ density matrices $= \{\rho \in \mathcal{T}(\mathcal{H}) : \rho \geq 0,\ Tr(\rho) = 1\}$

Tr is the non-normalized trace and tr is a tracial state.

Von Neumann algebras $\mathcal{A} \subset \mathcal{B}(\mathcal{G}),\ \mathcal{B} \subset \mathcal{B}(\mathcal{H}),\ \mathcal{C} \subset \mathcal{B}(\mathcal{K})$ with

normal states ϕ on $\mathcal{A}$ or $\mathcal{B}$, ψ on $\mathcal{C}$.

Note: Because $\mathcal{H}$ is separable, the predual $\mathcal{A}_*$ of $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is separable and there exists a faithful normal state for $\mathcal{A}$, see [Sa71], 2.1.9 and 2.1.10.

By 'stochastic matrix' we mean a matrix with non-negative entries such that all the row sums equal one.

We use the term 'stochastic map' as abbreviation for 'normal unital completely positive map': S, T (compare also A.1),

in particular $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$.

$\mathcal{Z}$ denotes a certain set of stochastic maps, see 1.2.1.

$S : (\mathcal{A}, \phi_{\mathcal{A}}) \to (\mathcal{B}, \phi_{\mathcal{B}})$ means that the stochastic map S maps $\mathcal{A}$ into $\mathcal{B}$ and respects the states $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$ in the sense that $\phi_{\mathcal{B}} \circ S = \phi_{\mathcal{A}}$.

Preadjoints of stochastic maps: C, $D, \ldots$

Homomorphism of a von Neumann algebra always means a (not necessarily unital) normal *−homomorphism: j, J

Unital endomorphisms: α

Conditional expectations: P, Q

If $w : \mathcal{G} \to \mathcal{H}$ is a linear operator, then we write $Ad\,w = w \cdot w^* : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$, even if w is not unitary.

General references for operator algebras are [Sa71, Ta79, KR83].

Probability spaces: (Ω, Σ, μ)

$\mathcal{M}(p, q)$ are the joint probability distributions for measures p, q and $\mathcal{S}(q, p)$ are the transition operators S with $p \circ S = q$, see Section 4.1.

Larger objects often get a tilde $\tilde{}$ or hat $\hat{}$, for example $\tilde{\mathcal{A}}$.

This should help to get a quick orientation but of course the conventions may be violated in specific situations and the reader has to look for the definition in the main text. We have made an attempt to invent a scheme of notation which provides a bridge between different chapters and sections and stick to it even if it is more clumsy than it would have been possible if the parts had been treated in isolation. We think that the advantages are more important. Besides the quick orientation already mentioned, the reader can grasp connections in this way even before they are explicitly formulated. Nevertheless, there is a moderate amount of repetition of definitions if the same occurs in different chapters to make independent reading easier.

Numbering of chapters, sections and subsections is done in the usual way. Theorems, propositions, lemmas etc. do not get their own numbers but are cross-referenced by the number of the subsection in which they are contained.

1

Extensions and Dilations

In this chapter we are concerned with normal unital completely positive maps on von Neumann algebras which we call 'stochastic maps' for short. In the context of noncommutative stochastic processes, such maps play the role of transition operators and therefore they deserve a careful study on their own. To come as directly as possible to the heart of the matter and to new results, we have postponed a review of the basic structure theory of such maps to the Appendix (in particular A.1 and A.2) which can be used by the reader according to individual needs.

Our first topic is an extension problem for stochastic maps which occurs naturally in connection with the GNS-construction. For concreteness, we start with some elementary computations in the easiest nontrivial case and then discuss the general case. Extension problems for completely positive maps are first discussed by W. Arveson in [Ar69], see also [ER00] for a recent survey. The additional point which we make consists in the inclusion of states. This is well motivated from the probabilistic point of view. Our main observation here is a duality between this extension problem and a dilation problem. The latter is described under the heading 'weak tensor dilations' of stochastic maps. While this is new as an explicit concept, it should be compared especially to some early dilation theories of completely positive maps, for example by D.E. Evans and J.T. Lewis [EL77], E.B. Davies [Da78], G.F. Vincent-Smith [Vi84]. It differs from them by insisting on a tensor product structure.

We define an equivalence relation for weak tensor dilations which later turns out to be the correct one in the duality with the extension problem. To formulate the correspondence between extension and dilation, we have to consider commutants and the dual stochastic map on the commutants. Then under specified conditions every solution of the extension problem gives rise to an equivalence class of solutions of the dilation problem and conversely. Thus we see that these problems shed some light onto each other. The remainder of the chapter discusses further details of this correspondence, instructive special cases and further examples. Applications based on the fact that stochastic maps can be interpreted as transition operators in noncommutative probability are postponed to the following chapters. The correspondence then proves to be a useful tool. Parts of the contents of Chapter 1 are also discussed in [Go1].

1.1 An Example with 2 × 2 - Matrices

1.1.1 A Stochastic Map

Consider a stochastic matrix $\begin{pmatrix} 1-\lambda & \lambda \\ \mu & 1-\mu \end{pmatrix}$ with $0 < \lambda, \mu < 1$.
We think of it as an operator

$$S : \mathbb{C}^2 \to \mathbb{C}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} (1-\lambda)x_1 + \lambda x_2 \\ \mu x_1 + (1-\mu)x_2 \end{pmatrix}.$$

S is a **stochastic map**, which means here that it is positive and unital. See A.1 for the general definition and further discussion. There is an invariant probability measure $\phi = \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu}\right)$ in the sense that $\phi \circ S = \phi$.

We can apply the GNS-construction for the algebra $\mathcal{A} = \mathbb{C}^2$ with respect to the state ϕ, and we get the Hilbert space $\mathcal{H} = \mathbb{C}^2$ (with canonical scalar product) and the unit vector $\Omega = \frac{1}{\sqrt{\lambda+\mu}} \begin{pmatrix} \sqrt{\mu} \\ \sqrt{\lambda} \end{pmatrix}$. Now the state ϕ is realized as a vector state, i.e. $\phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \langle \Omega, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \Omega \rangle$. Identifying $\mathcal{A} = \mathbb{C}^2$ with the diagonal subalgebra of $\mathcal{B}(\mathcal{H}) = M_2$ we have

$$S : \quad x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mapsto \begin{pmatrix} (1-\lambda)x_1 + \lambda x_2 & 0 \\ 0 & \mu x_1 + (1-\mu)x_2 \end{pmatrix}$$

and $\langle \Omega, x\Omega \rangle = \langle \Omega, S(x)\Omega \rangle$. We shall now ask for stochastic maps $Z : M_2 \to M_2$ (i.e. unital completely positive maps, see A.1.1) which extend S and satisfy $\langle \Omega, x\Omega \rangle = \langle \Omega, Z(x)\Omega \rangle$ for all $x \in M_2$.

1.1.2 Direct Approach

Let us try a direct approach. Any completely positive map $Z : M_2 \to M_2$ can be written in the form $Z(x) = \sum_{k=1}^{d} a_k \, x \, a_k^*$ with $a_k \in M_2$. Introduce four vectors $a_{ij} \in \mathbb{C}^d$, $i, j = 1, 2$, whose k-th entry is the ij-entry of a_k (compare [Kü85b]). With the canonical scalar product and euclidean norm on $\mathbb{C}^d$ we get by direct computation:

$$Z \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} \|a_{11}\|^2 x_1 + \|a_{12}\|^2 x_2 & \langle a_{21}, a_{11} \rangle x_1 + \langle a_{22}, a_{12} \rangle x_2 \\ \langle a_{11}, a_{21} \rangle x_1 + \langle a_{12}, a_{22} \rangle x_2 & \|a_{21}\|^2 x_1 + \|a_{22}\|^2 x_2 \end{pmatrix}$$

If Z is an extension of S we conclude that