René Vidal
Anders Heyden
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LNCS 4358

Dynamical Vision

ICCV 2005 and ECCV 2006 Workshops
WDV 2005 and WDV 2006
Beijing, China, October 2005
Graz, Austria, May 2006, Revised Papers

Springer

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Library of Congress Control Number: 2007920190

CR Subject Classification (1998): I.4, I.2.10, I.5, I.3, H.5.2-3

LNCS Sublibrary: SL 6 – Image Processing, Computer Vision, Pattern Recognition, and Graphics

ISSN 0302-9743
ISBN-10 3-540-70931-2 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-70931-2 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media

springer.com

Printed in Germany

Typesetting: Camera-ready by author, data conversion by Scientific Publishing Services, Chennai, India
Printed on acid-free paper SPIN: 12021055 06/3142 5 4 3 2 1 0

Lecture Notes in Computer Science 4358

Commenced Publication in 1973
Founding and Former Series Editors:
Gerhard Goos, Juris Hartmanis, and Jan van Leeuwen

Lecture Notes in Computer Science

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Preface

Classical multiple-view geometry studies the reconstruction of a static scene observed by a rigidly moving camera. However, in many real-world applications the scene may undergo much more complex dynamical changes. For instance, the scene may consist of multiple moving objects (e.g., a traffic scene) or articulated motions (e.g., a walking human) or even non-rigid dynamics (e.g., smoke, fire, or a waterfall). In addition, some applications may require interaction with the scene through a dynamical system (e.g., vision-guided robot navigation and coordination).

To study the problem of reconstructing *dynamical scenes*, many new algebraic, geometric, statistical, and computational tools have recently emerged in computer vision, computer graphics, image processing, and vision-based control. The goal of the International Workshop on Dynamical Vision (WDV) is to converge different aspects of the research on dynamical vision and to identify common mathematical problems, models, and methods for future research in this emerging and active area.

This book reports 24 contributions presented at the First and Second International Workshops on Dynamical Vision, WDV 2005 and WDV 2006, which were held in conjunction with the 10th International Conference on Computer Vision (ICCV 2005) and the 9th European Conference on Computer Vision (ECCV 2006), respectively. These contributions were selected from over 52 submissions through a rigorous double-blind review process by members of the Program Committee. The book is structured in six parts, each containing three to five contributions on six topics of dynamical vision: (1) motion segmentation and estimation, (2) human motion analysis, tracking and recognition, (3) dynamic textures, (4) motion tracking, (5) rigid and non-rigid motion analysis, and (6) motion filtering and vision-based control.

The success of these workshops would not have been possible without the outstanding quality of reviews by members of the Program Committee, the financial support provided by several sponsors, and the technical support provided by Avinash Ravichandran of The Johns Hopkins University.

October 2006

René Vidal
Anders Heyden
Yi Ma

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The Space of Multibody Fundamental Matrices: Rank, Geometry and Projection

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Abstract. We study the rank and geometry of the multibody fundamental matrix, a geometric entity characterizing the two-view geometry of dynamic scenes consisting of multiple rigid-body motions. We derive an upper bound on the rank of the multibody fundamental matrix that depends on the number of independent translations. We also derive an algebraic characterization of the SVD of a multibody fundamental matrix in the case of two or odd number of rigid-body motions with a common rotation. This characterization allows us to project an arbitrary matrix onto the space of multibody fundamental matrices using linear algebraic techniques.

1 Introduction

Given two perspective views of a scene containing multiple rigidly moving objects, we consider the problem of estimating the motion of each object relative to the camera, without knowing which measurements belong to which object.

When the scene is *static*, i.e., when either the camera or a single object move rigidly, it is well-known [7] that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{P}^2$ are two perspective images of a point in 3-D space, then they must satisfy the *epipolar constraint*

$$\boldsymbol{x}_2^\top F \boldsymbol{x}_1 = 0, \tag{1}$$

where $F \in \mathbb{R}^{3\times 3}$ is a rank-2 matrix called the *fundamental matrix*. The epipolar constraint can be used to estimate F and the camera motion from a set of point correspondences using linear techniques such as the eight-point algorithm. In the case of a calibrated camera, it is also known that F factors as $F = [T]_\times R$, where $[T]_\times \in so(3)$ is a skew-symmetric matrix associated with the camera translation $T \in \mathbb{R}^3$ and $R \in SO(3)$ is the camera rotation. The space $so(3) \times SO(3)$ is known as the *essential manifold* and can be characterized as the space of matrices with singular values $\{\|T\|, \|T\|, 0\}$. Such a characterization is crucial when estimating F from noisy correspondences, because it allows us to project a noisy linear estimate of F onto a geometrically correct *essential matrix*.

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The work of [14] proposes a generalization of the eight-point algorithm to the more general and challenging case of *dynamic scenes* in which both the camera and an unknown number of objects with unknown 3-D structure move independently. The paper shows that applying a polynomial embedding to the image points leads to the so-called *multibody epipolar constraint* and its associated *multibody fundamental matrix* $\mathcal{F}$. The method computes the number of motions from a rank constraint on the image measurements, estimates the multibody fundamental matrix using least squares, and the individual fundamental matrices using multivariate polynomial factorization or differentiation.

Unfortunately, the method is not yet reliable in the presence of noise, because of the following reasons:

1. The polynomial embedding is not invariant with respect to rotations or translations of the image data, which makes it difficult to characterize the space of multibody fundamental matrices. Such a characterization is crucial for improving the performance of linear algorithms in the presence of noisy data.
2. The multibody fundamental matrix $\mathcal{F}$ is computed *linearly*, without taking into account nonlinear constraints dictated by its rank and geometry. Therefore, the estimate of $\mathcal{F}$ may not be geometrically correct in the presence of noise, meaning that it may not perfectly factor into the multiple fundamental matrices associated with each one of the rigid-body motions.

In this paper, we show how to overcome these difficulties by exploiting the rank and geometry of the multibody fundamental matrix. More specifically,

1. **Rank:** we show that the rank of $\mathcal{F}$ depends on the number of independent translational motions and on the number of times they are repeated. Our results complete the analysis in [14], which deals with the particular case of one repeated translational motion.
2. **Geometry:** we show that in the case of n rigid-body motions with common rotation, $\mathcal{F}$ factors as the product of a symmetric (n even) or skew-symmetric (n odd) matrix times a rotation matrix. When the number of motions is two or odd, this leads to a characterization of the SVD of $\mathcal{F}$. This characterization is possible thanks to a slightly new definition of the polynomial embedding that makes the singular values of the multibody fundamental matrix invariant with respect to rotations of the image data.
3. **Projection:** we show that the characterization of the SVD of $\mathcal{F}$ can be used to project an arbitrary matrix estimated from noisy correspondences onto the space of multibody fundamental matrices using linear algebraic techniques.

To the best of our knowledge, there is no prior work studying the geometry and projection onto the space of multibody fundamental matrices. In fact, finding a linear algebraic characterization of this space is an extremely challenging problem. Therefore, although the case of two or odd number of motions with common rotations may appear to be restrictive, we believe this case is an important step toward solving the general case.

Previous work. Most prior work on dynamic scene reconstruction proceeds by first segmenting image measurements into various motion models, and then estimating a single motion model for each group of measurements, or else in an iterative manner with the aid of the EM algorithm. The number of models can also be estimated in a probabilistic framework using model selection techniques such as [10,6]. However, the convergence of iterative/probabilistic methods to the global optimum depends strongly on correct initialization [10,9]. This has motivated the recent development of geometric approaches to dynamic scene reconstruction which do not require initialization. Algebraic approaches include methods for multiple moving objects seen by an orthographic camera [1,5,17,11], self-calibration from multiple motions [2], multiple points moving in planes [8], segmentation of two [16] and multiple [14,15] rigid-body motions from two or three [4] perspective views.

2 Multibody Epipolar Geometry

Given a set of point correspondences $\{(\boldsymbol{x}_1^j, \boldsymbol{x}_2^j)\}_{j=1}^N$ generated from n independently and rigidly moving objects, our goal is to estimate their associated fundamental matrices $\{F_i\}_{i=1}^n$ and the object to which each image pair belongs.

To this end, let $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ be an arbitrary image pair associated with *any* of the n moving objects. Then, there exists a fundamental matrix $F_i \in \mathbb{R}^{3\times 3}$ such that the *epipolar constraint* $\boldsymbol{x}_2^\top F_i \boldsymbol{x}_1 = 0$ is satisfied. Therefore, regardless of the object associated with the image pair, the following *multibody epipolar constraint* [14] must be satisfied by the fundamental matrices $\{F_i\}_{i=1}^n$ and the image pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$

$$\mathrm{MEC}(\boldsymbol{x}_1, \boldsymbol{x}_2) \doteq \prod_{i=1}^{n} \left(\boldsymbol{x}_2^\top F_i \boldsymbol{x}_1\right) = 0. \tag{2}$$

The multibody epipolar constraint (MEC) is a homogeneous polynomial of degree n in each of $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$. Therefore, if we let $\boldsymbol{x}_1 = [x_1, y_1, z_1]^\top$, equation (2) viewed as a function of $\boldsymbol{x}_1$ can be written as a linear combination of the following $M_n \doteq (n+1)(n+2)/2$ independent monomials $\{x_1^n, x_1^{n-1}y_1, x_1^{n-1}z_1, \ldots, z_1^n\}$. After collecting all these monomials into a vector

$$\nu_n(\boldsymbol{x}_1) = [\ldots, \gamma_{n_1,n_2,n_3} x_1^{n_1} y_1^{n_2} z_1^{n_3}, \ldots]^\top \in \mathbb{R}^{M_n}, \tag{3}$$

where $\gamma_{n_1,n_2,n_3} = \sqrt{\frac{n!}{n_1!n_2!n_3!}}$ with $0 \le n_1, n_2, n_3 \le n$, $n_1 + n_2 + n_3 = n$, the MEC can be written as the following a bilinear expression in $\nu_n(\boldsymbol{x}_1)$ and $\nu_n(\boldsymbol{x}_2)$ (see [14]):

$$\nu_n(\boldsymbol{x}_2)^\top \mathcal{F} \nu_n(\boldsymbol{x}_1) = 0. \tag{4}$$

The matrix $\mathcal{F} \in \mathbb{R}^{M_n \times M_n}$ is called the *multibody fundamental matrix*, and is a natural generalization of the fundamental matrix $F \in \mathbb{R}^{3\times 3}$ to the case of n moving objects. The embedding $\nu_n : \mathbb{R}^3 \to \mathbb{R}^{M_n}$ is known in algebraic geometry as the Veronese map of degree n [3].

Remark 1 (Rotation invariant). Notice that our definition of the Veronese map is slightly different from the one in [14], as we deliberately multiply the monomial $x_1^{n_1} y_1^{n_2} z_1^{n_3}$ by the coefficient γ_{n_1,n_2,n_3}. As we will show in Theorem 2, this new definition of the Veronese map makes it rotation invariant, a property that will be shown to be crucial for characterizing the space of multibody fundamental matrices.

Thanks to the Veronese map, we can write the epipolar constraint for all N point correspondences as

$$\boldsymbol{V}_n \boldsymbol{f} \doteq \left[\nu_n(\boldsymbol{x}_2^1) \otimes \nu_n(\boldsymbol{x}_1^1) \cdots \nu_n(\boldsymbol{x}_2^N) \otimes \nu_n(\boldsymbol{x}_1^N)\right]^\top \boldsymbol{f} = \boldsymbol{0}, \tag{5}$$

where $\boldsymbol{f} \in \mathbb{R}^{M_n^2}$ is the stack of the rows of $\mathcal{F}$ and $\otimes$ represents the Kronecker product. Given $\mathcal{F}$, which can be computed as the least squares solution of (5), the individual fundamental matrices $\{F_i\}_{i=1}^n$ are obtained by factorizing the bi-homogeneous polynomial

$$\nu_n(\boldsymbol{x}_2)^\top \mathcal{F} \nu_n(\boldsymbol{x}_1) = \prod_{i=1}^{n} \left(\boldsymbol{x}_2^\top F_i \boldsymbol{x}_1\right) = 0. \tag{6}$$

into a product of bilinear forms [14], or from the second order derivatives of the MEC [12].

Notice that the multibody fundamental matrix $\mathcal{F}$ is determined by the fundamental matrices of the individual rigid motions $\{F_i\}_{i=1}^n$. Since these fundamental matrices are of rank two and/or belong to the essential manifold, the multibody fundamental matrix is not an arbitrary matrix in $\mathbb{R}^{M_n \times M_n}$, but must satisfy some nonlinear constraints, such as rank constraints and/or geometric constraints. Such constraints are clearly not exploited by the linear algorithm of [14]. Therefore, the linear estimate of the multibody fundamental matrix may not be geometrically correct in the presence of noise, meaning that its associated MEC may not perfectly factor as a product of epipolar constraints.

Such problems motivate our development in the rest of this paper.

3 Rank of the Multibody Fundamental Matrix

It is well-known [7] that the rank of a fundamental matrix F is two. The vector $\boldsymbol{e}$ in its left null space is called the *epipole* and satisfies the following relationship $\boldsymbol{e}^\top F = 0$.

In the case of n rigid-body motions, there exist n epipoles $\{\boldsymbol{e}_i\}_{i=1}^n$ such that $\boldsymbol{e}_i^\top F_i = 0$. This implies that

$$\left(\boldsymbol{e}_i^\top F_1 \boldsymbol{x}\right)\left(\boldsymbol{e}_i^\top F_2 \boldsymbol{x}\right) \cdots \left(\boldsymbol{e}_i^\top F_n \boldsymbol{x}\right) = \nu_n(\boldsymbol{e}_i)^\top \mathcal{F} \nu_n(\boldsymbol{x}) = 0, \tag{7}$$

for all $\boldsymbol{x} \in \mathbb{P}^2$. Since the vector $\nu_n(\boldsymbol{x})$ spans all of $\mathbb{R}^{M_n}$ when $\boldsymbol{x}$ ranges over $\mathbb{P}^2$,[1] we immediately have [14]

$$\nu_n(\boldsymbol{e}_i)^\top \mathcal{F} = 0 \quad \text{for} \quad i = 1, \ldots, n. \tag{8}$$

[1] This is simply because the M_n monomials in $\nu_n(\boldsymbol{x})$ are linearly independent.

Therefore, the multibody fundamental matrix $\mathcal{F}$ is also rank deficient, because the n *embedded epipoles* $\{\nu_n(\boldsymbol{e}_i)\}_{i=1}^n$ lie in its left null space. Notice, however, that the dimension of the null space of $\mathcal{F}$ need not be n, because the embedded epipoles may not be linearly independent. For instance, if two different rigid-body motions have the same translation, but different rotation, then they have the same epipole, hence the same embedded epipole.

The purpose of this section is to characterize the null space of $\mathcal{F}$ as a function of the number of motions n, the number of different epipoles $n_e \leq n$ (different up to a scale factor) and the number of times $\{k_i\}_{i=1}^{n_e}$, with $\sum_{i=1}^{n_e} k_i = n$, that each epipole is repeated.[2] More specifically, we prove the following theorem.

Theorem 1 (Null space of $\mathcal{F}$). *Let $\mathcal{F}$ be the multibody fundamental matrix generated by n fundamental matrices. Let n_e be the number of different epipoles and k_i, $i = 1, \ldots, n_e$, be the number of times each different epipole is repeated. The rank of the multibody fundamental matrix is bounded by*

$$\operatorname{rank}(\mathcal{F}) \leq M_n - \sum_{i=1}^{n_e} M_{k_i-1} \leq M_n - n, \tag{9}$$

where the inequality on the right hand side is true regardless of whether the epipoles are repeated or not.

The formal proof of the theorem is organized as follows. In Section 3.1, we show that if an epipole $\boldsymbol{e}_i$ is repeated k_i times, then all the derivatives of ν_n of order less than k_i evaluated at $\boldsymbol{e}_i$ lie in the left null space of $\mathcal{F}$. In Section 3.2, we show that only M_{k_i-1} of these derivatives are linearly independent, thus each different epipole contributes with an M_{k_i-1}-dimensional subspace to null$(\mathcal{F})$. In Section 3.3 we show that these n_e subspaces are independent, meaning that they intersect only at $\mathbf{0}$. Therefore, the dimensionality of the null space of $\mathcal{F}$ is at least $\sum_{i=1}^{n_e} M_{k_i-1} \geq n$.

3.1 Partial Derivatives at Repeated Epipoles

In this subsection, we show that when an epipole $\boldsymbol{e}_i$ is repeated k_i times, not only $\nu_n(\boldsymbol{e}_i)$ is in the null space of $\mathcal{F}$, as shown by equation (8), but also the derivatives of $\nu_n(\boldsymbol{x})$ of order less than k_i at $\boldsymbol{e}_i$. Before proving this, we need the following technical lemma, which allows us to express the derivatives of the nth order MEC as a linear combination of MECs of lower order.

Lemma 1. *Let $\mathcal{F}^{(n)}$ be the multibody fundamental matrix generated by $F_1, \ldots, F_n$. Let $\mathcal{F}_j^{(n-l)}$ be a multibody fundamental matrix generated by a choice of $n-l$ out of the n fundamental matrices for $j = 1, \ldots, \binom{n}{l}$. Then $\forall (l_1, l_2, l_3)$, such that $l_1 + l_2 + l_3 = l$, $\forall \boldsymbol{x} = [x, y, z]^\top$, $\forall \boldsymbol{y} \in \mathbb{P}^2$, we have*

[2] The particular case in which one epipole is repeated k times, and the other $n-k$ epipoles are different can be found in [14].

$$\frac{\partial^l(\nu_n(\boldsymbol{x})^\top \mathcal{F}^{(n)} \nu_n(\boldsymbol{y}))}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}} = \sum_{j=1}^{\binom{n}{l}} \alpha_j \nu_{n-l}(\boldsymbol{x})^\top \mathcal{F}_j^{(n-l)} \nu_{n-l}(\boldsymbol{y}), \tag{10}$$

where the coefficient $\alpha_j \in \mathbb{R}$ depends on $\mathcal{F}^{(n)}$ and $\boldsymbol{y}$, but is independent of $\boldsymbol{x}$.

We are now ready to show that the derivatives of ν_n at a repeated epipole lie in the left null space of $\mathcal{F}$.

Lemma 2. *If $\boldsymbol{e}_i \in \mathbb{P}^2$ is an epipole that is repeated k_i times, and $\boldsymbol{x} = [x, y, z]^\top$, then $\forall (l_1, l_2, l_3)$, such that $l_1 + l_2 + l_3 = l \leq k_i - 1$, we have*

$$\left. \frac{\partial^l \nu_n(\boldsymbol{x})^\top}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}} \right|_{\boldsymbol{e}_i} \mathcal{F} = \mathbf{0}. \tag{11}$$

Proof. Since $\boldsymbol{e}_i$ is repeated k_i times, there are k_i fundamental matrices whose left null space is $\boldsymbol{e}_i$. Then any choice of $n-l$ fundamental matrices with $l \leq k_i - 1$ will contain at least one fundamental matrix whose left null space is $\boldsymbol{e}_i$. From (8) we have that $\boldsymbol{e}_i$ is an epipole for each one of the multibody fundamental matrices $\mathcal{F}_j^{(n-l)}$ with $l \leq k_i - 1$, i.e., $\nu_{n-l}(\boldsymbol{e}_i)^\top \mathcal{F}_j^{(n-l)} = \mathbf{0}$. This, together with Lemma 1, implies that for all $\boldsymbol{y} \in \mathbb{P}^2$ and for all (l_1, l_2, l_3) such that $l_1 + l_2 + l_3 = l \leq k_i - 1$

$$\left. \frac{\partial^l \nu_n(\boldsymbol{x})^\top}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}} \right|_{\boldsymbol{e}_i} \mathcal{F} \nu_n(\boldsymbol{y}) = \mathbf{0}.$$

Since this is true for all $\boldsymbol{y} \in \mathbb{P}^2$, the claim follows.

3.2 Dimension of the Subspaces Spanned by the Partial Derivatives

In this subsection, we show that an epipole repeated k_i times contributes to the null space of $\mathcal{F}$ with a subspace of dimension at least M_{k_i-1}. The result is a consequence of the following facts: 1) the subspace spanned by the partial derivatives of order l is included in any of the subspaces spanned by higher order partial derivatives; and 2) the dimension of the subspace spanned by the derivatives of order l is M_l.

First, notice that each entry of $\nu_n(\boldsymbol{x})$ is of the form $\gamma_{n_1,n_2,n_3} x^{n_1} y^{n_2} z^{n_3}$ with $n_1 + n_2 + n_3 = n$. After some simple algebraic calculations, we can show that

$$(n-l)\frac{\partial^l \nu_n(\boldsymbol{x})}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}} = [\frac{\partial^{l+1} \nu_n(\boldsymbol{x})}{\partial x^{l_1+1} \partial y^{l_2} \partial z^{l_3}}, \frac{\partial^{l+1} \nu_n(\boldsymbol{x})}{\partial x^{l_1} \partial y^{l_2+1} \partial z^{l_3}}, \frac{\partial^{l+1} \nu_n(\boldsymbol{x})}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3+1}}]\boldsymbol{x}. \tag{12}$$

Therefore, if we let $A_l(\boldsymbol{x})$ be the span of the l-th order partial derivatives of $\nu_n(\boldsymbol{x})$, then (12) implies that $A_l(\boldsymbol{x}) \subseteq A_{l+1}(\boldsymbol{x})$ for all $0 \leq l < n$. By simple induction we have that if $\boldsymbol{e}_i$ is an epipole that is repeated k_i times, then

$$A_0(\boldsymbol{e}_i) \subseteq A_1(\boldsymbol{e}_i) \subseteq \cdots \subseteq A_{k_i-1}(\boldsymbol{e}_i). \tag{13}$$

As a consequence of (13), studying the dimension of the subspace spanned by all the partial derivatives at a repeated epipole *up to* a certain order, boils down