BISHWANATH CHAKRABORTY

PRINCIPLES OF PLASMA MECHANICS

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BISHWANATH CHAKRABORTY
Jadavpur University, Calcutta
and
Calcutta University, Agartala Centre

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PREFACE

This book deals with the essentials of theoretical plasma mechanics and magnetohydrodynamics (MHD). It has evolved from lectures I delivered to the senior students of the mathematics department of Jadavpur University. When the course was first introduced, in 1967, I was given the responsibility of framing the curriculum and teaching it right from the very outset. I found to my chagrin that the available textbooks were of such an advanced level that they far exceeded the students ability to understand the subject. As such, I had to make copious notes from various available sources for my lectures, keeping the students difficulties in mind. During the past ten years much new material has been added and the notes have been revised and rewritten, resulting in this book.

The first five chapters of this book more or less cover the needs of an introductory course. The remaining seven chapters are additions of a somewhat advanced level. Their inclusion, I hope, will increase the utility of the book, as, together with the first five chapters, they can be used as a complete advanced level course of theoretical plasma physics, leading to a research programme on the subject.

In addition to the twelve chapters, the book has some appendices. A few of these appear at the end of the book and some, of a mathematical nature, appear at the end of chapters where they become necessary for clarifying mathematical concepts.

Contrary to present day convention I have introduced gaussian or cgs units throughout the book. The choice is due not entirely to personal habit, the available literature is also in gaussian units. The metric units have yet to make their impact on plasma research. I, therefore, request my readers to bear with me in the choice of units.

In every chapter, many references, particularly those which have been consulted, are given. However, no special effort has been made to cite all the useful and standard references on the subject.

My contact with a number of mathematicians and physicists, especially Professors A.A. Vlasov, J.N. Kapur, P.C. Jain, P.K. Ghosh, N.R. Sen, B.B. Sen, B.S. Ray, M. Dutta, H. Bremmer, L.J.F. Broer, F.W. Sluijter and L. Stenflo has greatly influenced my understanding of mathematics

and physics. I take this opportunity to express my indebtedness to them.

My inexpressible gratitude, however, remains for Shankar Brahmachari —the Guru—who inspired me to work on this book.

I gratefully acknowledge and remember the kind services of the staff of the libraries of Jadavpur University, Calcutta, Indian Association for the Cultivation of Science, Calcutta, Physics Faculty of Moscow State University, University of Technology of Eindhoven and Umea University.

I also wish to thank the National Book Trust, India, for subsidizing publication of the book and the University Grants Commission for providing financial help to prepare the manuscript. The authorities of Jadavpur University have always been very helpful to me during my work on the book. Professors Rajat Kumar Chakravarty and Debiprasad Chattopadhyaya have greatly assisted me in the publication of this book. I express my sincere thanks to them.

It is a pleasure for me to express my gratitude to my family, more especially to my wife, Manju, who cooperated in every way during my work on the book.

If this book could serve as an incentive to some readers for a deeper study of some plasma problems my expectation would largely be fulfilled.

Calcutta
October 3, 1978

BISHWANATH CHAKRABORTY

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Chapter One

BASIC ELECTRICITY AND MAGNETISM

1. Introduction

The fundamental laws of classical electricity and magnetism are formulated by experimentally studying the behaviour of its basic elements, that is, the electric charges and magnetic dipoles. Based on these laws, the basic equations of electrodynamics, known as Maxwell equations, can be deduced. Moreover, based on laboratory experiments, the expression for the force exerted by these elements among themselves, called the Lorentz force, is determined.

Considering only electrical properties all bodies can be regarded as dielectrics or conductors. A dielectric material is composed of aggregates of charges which are normally bound to positions of equilibrium by internal forces. An electric dipole is defined as two equal but opposite charges which are a very small distance apart and which are bound together by the action of the internal molecular forces. So in dielectrics a volume distribution of dipoles is formed.

A conductor, by definition, is a substance in which charges are free to move under the action of an applied electric field through distances which are large compared to atomic dimensions. This movement occurs even when the electric fields are very small, and give rise to the concept of electrical conductivity of matter. The current density and electric field intensity are usually found to be proportional in conductors.

If, in a volume τ, there are N charges at N positions, and if the position vector of the ith charge e_i at the time t be $\mathbf{r}_i(t)$ ($i = 1, 2, \ldots, N$), then the charge density ρ and the current density $\mathbf{j}$ are defined as

$$\rho = \frac{1}{\tau} \sum_{i=1}^{N} e_i \, \delta(\mathbf{r} - \mathbf{r}_i(t)), \tag{1.1}$$

$$\mathbf{j} = \frac{1}{\tau} \sum_{i=1}^{N} e_i \, \mathbf{u}_i \, \delta(\mathbf{r} - \mathbf{r}_i(t)) \tag{1.2}$$

where $\mathbf{u}_i$ is the velocity of the ith particle at the position $\mathbf{r} = \mathbf{r}_i$, δ is the delta function of Dirac (see Appendix 4) and a dot upon a quantity means its derivative with respect to time.

Since in dielectrics a negative charge is associated with a positive charge to form a dipole, an applied electric field brings about a redistribution of the average charge density ρ_i, without changing the total number of charges. So

$$\iiint_{\tau} \rho_i \, d\tau = 0$$

where τ is the volume of the dielectric. Since this integral equation is valid for a body of arbitrary shape, ρ_i can be written as the divergence of a vector; thus

$$\rho_i = -\operatorname{div} \mathbf{P}, \iiint_{\tau} \rho_i \, d\tau = 0 = -\iiint_{\tau} \operatorname{div} \mathbf{P} \, d\tau \tag{1.3}$$

P is called dielectric polarization vector, or simply, the polarization and the body in such cases is said to be polarized by the electric field.

Applying Green's theorem (see Appendix 5) on the right hand side we get

$$0 = -\iint_{\Sigma} P_N \, d\Sigma$$

where P_N is the component of **P** along the normal to the surface Σ of volume τ. Since this relation is, in general, true, $P_N = 0$ at every point of Σ. For finding the physical significance of **P** the dipole moment density $\mathbf{r}\rho_i$ is integrated over the whole volume τ. The result is

$$\iiint_{\tau} \mathbf{r} \rho_i \, d\tau = -\iiint_{\tau} \mathbf{r} \operatorname{div} \mathbf{P} \, d\tau = -\iint_{\Sigma} \mathbf{r} (\mathbf{P} \cdot d\boldsymbol{\Sigma}) + \iiint_{\tau} \mathbf{P} \, d\tau$$

Since P_N is zero at every point of Σ we can write

$$\iiint_{\tau} \mathbf{r} \rho_i \, d\tau = \iiint_{\tau} \mathbf{P} \, d\tau$$

Hence the polarization vector **P** is the dipole moment per unit volume. If, in addition, charges not belonging to a dielectric are brought from outside τ and their density is ρ_e then the total charge density $\rho = \rho_e + \rho_i$.

A volume of plasma (to be defined and discussed in detail in Chapter 3) can be regarded as a cloud of mobile positive and negative charges and so has both electrical conductivity and dielectric permittivity. Charges in a plasma being free to a certain extent, mutual exchange of momentum, through collisions, imparting a finite amount of electrical conductivity (or resistance) to the medium is possible. On the other hand, charges being bound together to an extent, characteristic small amplitude oscillations of them are possible if the medium is disturbed in such a manner that a high degree of charge neutrality i. maintained.

In this chapter, the fundamental laws will be briefly described and the basic equations deduced from them. Some aspects of electrodynamics, which are relevant to plasmas, will also be discussed.

2. Static and dynamic laws

1. We first consider an electric charge e placed at the point O. Let P be a point in space on the surface element $d\Sigma$ where the unit charge is kept to study the effects of the charge e at O.

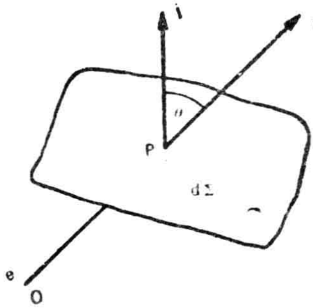

Figure 2.1 The field across the surface element $d\Sigma$ at P due to the charge e at O.

Let $\mathbf{i}(= \mathbf{OP}/OP)$ be the unit vector along **OP** and **j** be the unit vector normal to $d\Sigma$ at P in such a sense that **j** coincides with the outward drawn normal to a closed surface which contains $d\Sigma$. The electric field at P is $\mathbf{E} = \mathbf{i}e/r^2$ such that

$$(\mathbf{E}\cdot d\mathbf{\Sigma}) = \frac{e\, d\Sigma \cos\theta}{r^2} \tag{2.1}$$

where θ is the angle between **i** and **j**. If $d\Omega$ is the solid angle subtended at O by $d\Sigma$, then $(\mathbf{E}\cdot d\mathbf{\Sigma}) = e\, d\Omega$, because $r^2 d\Omega = d\Sigma \cos\theta$.

The point O can be inside, on or outside Σ. Hence we can write

$$\iint_\Sigma (\mathbf{E}\cdot d\mathbf{\Sigma}) = e\iint_\Sigma d\Omega = 4\pi e, \quad \text{if } O \text{ lies inside } \Sigma$$

$$= 2\pi e, \quad \text{if } O \text{ lies on } \Sigma$$

$$= 0, \quad \text{if } O \text{ lies outside } \Sigma$$

To avoid a discussion not relevant to our purpose, the case of the point O on Σ will be ignored.

When several charges are held fixed at known positions, let O_i be the position of the ith charge e_i and $O_iP = \mathbf{r}_i$. Then the force $\mathbf{E}$ on an unit charge at $P = \sum_i e_i \mathbf{r}_i / r_i$ and so

$$\iint_\Sigma (\mathbf{E} \cdot d\Sigma) = \iint_\Sigma \sum_i \frac{e_i (\mathbf{r}_i \cdot d\Sigma)}{r_i^2}$$

If θ_i is the angle between $\mathbf{r}_i$ and $d\mathbf{\Sigma}$, then

$$\iint_\Sigma (\mathbf{E} \cdot d\mathbf{\Sigma}) = \sum_i \iint_\Sigma \frac{e_i \cos\theta_i \, d\Sigma}{r_i^2} = \sum_i \iint_\Sigma e_i \, d\Omega_i$$

where $d\Omega_i$ is the solid angle subtended by $d\Sigma$ at O_i. Using, in the summation, the index p for charges inside Σ and the index q for charges outside Σ, we can write

$$\iint_\Sigma (\mathbf{E} \cdot d\mathbf{\Sigma}) = \sum_p e_p \iint_\Sigma d\Omega_p + \sum_q e_q \iint_\Sigma d\Omega_q$$

Since O_q, the seat of the charge e_q, is outside Σ, $\iint_\Sigma d\Omega_q = 0$.

If, similarly, O_p is the seat of e_p,

$$\iint_\Sigma d\Omega_p = 4\pi \text{ and so } \iint_\Sigma (\mathbf{E} \cdot d\mathbf{\Sigma}) = 4\pi \sum_p e_p.$$

In a volume distribution, if ρ is the charge density at the point O and $d\tau$ is a volume element surrounding O, then

$$\iint_\Sigma (\mathbf{E} \cdot d\mathbf{\Sigma}) = 4\pi \iiint_\tau \rho \, d\tau$$

where τ is that part of the charged volume which is inside Σ. This is the integral form of Gauss' law of normal flux. By Green's theorem

$$\iint_\Sigma (\mathbf{E} \cdot d\mathbf{\Sigma}) = \iiint_\tau \operatorname{div} \mathbf{E} \, d\tau, \text{ and so } \iiint_\tau (\operatorname{div} \mathbf{E} - 4\pi\rho) \, d\tau = 0.$$

The above equation holds good for any arbitrary volume, large or small. So the integrand must vanish at each point, and therefore,

$$\operatorname{div} \mathbf{E} = 4\pi\rho.$$

This is the differential form of Gauss's law.

The charge density ρ consists of the sum $\rho_e + \rho_i$, where ρ_e is the density of charges which are brought from outside and ρ_i is that due to a redistribution of charges inside Σ due to the electric field. Hence, putting $\rho_i = -\operatorname{div} \mathbf{P}$, we get $\operatorname{div} \mathbf{E} = 4\pi\rho_e - 4\pi \operatorname{div} \mathbf{P}$ and so

$$\text{div } \mathbf{D} = 4\pi\rho_e \tag{2.2}$$

where
$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \tag{2.3}$$

Dropping the subscript e we get for Gauss's law of normal flux

$$\text{div } \mathbf{D} = 4\pi\rho. \tag{2.4}$$

2. Since separation or isolation of magnetic monopoles is not permitted by nature, only equal number of north and south poles occur together in any volume, small or large. Hence,

$$\iint_\Sigma (\mathbf{H}\cdot d\mathbf{\Sigma}) = 0, \quad \iint_\Sigma (\mathbf{B}\cdot d\mathbf{\Sigma}) = 0 \tag{2.4a}$$

where $\mathbf{H}$ is the magnetic field and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, where $\mathbf{M}$ is called magnetization vector and is defined in the manner of $\mathbf{P}$. Applying Green's theorem on the above integral we get

$$\text{div } \mathbf{H} = 0, \text{ div } \mathbf{B} = 0, \text{ div } \mathbf{M} = 0 \tag{2.5}$$

The vectors **M**, **B** and **H** being divergence free are called solenoidal vectors.

Relations (2.4a) and (2.5) express the divergence law of normal flux for t'.e magnetic field.

3. From experiments Faraday observed that if a closed circuit moves across a magnetic field, or the magnetic field through the circuit changes in time, a current flows even though no batteries are present. In either case current lasts only as long as the circuit moves or the field changes. It makes no difference whether the magnetic field is caused by a permanent magnet or an electrical circuit. For a mathematical formulation of the effects, let a circuit C_0, bounded by the open surface Σ, be considered. The magnetic flux linking the circuit is given by

$$\Phi = \iint_\Sigma (\mathbf{B}\cdot d\mathbf{\Sigma})$$

The electromotive force $\mathcal{E}$ around the circuit is the line integral

$$\mathcal{E} = \oint_{C_0} (\mathbf{E}\cdot d\mathbf{l})$$

where $\mathbf{E}$ is the electric intensity and $d\mathbf{l}$ is the line element at P along the circuit C_0.

Experimental results satisfy the following two laws:

(a) *If Φ is changed in any way then $\mathcal{E}$ is proportional to $d\Phi/dt$.*

(b) *This $\mathcal{E}$ acts in such a way as to oppose the change in Φ, that is, it acts so as to cause currents in the coil such that its magnetic effect can counteract the external change, and so $\mathcal{E}/(d\Phi/dt)$ is a negative quantity.*

Mathematically, these laws are contained in the equation

$$\mathcal{E} = -K\frac{d\Phi}{dt} \tag{2.6}$$

where K is the constant of proportionality. This is the integral form of Faraday's law of electromagnetic induction. If the line integral of $\mathcal{E}$ is transformed into a surface integral by Stoke's theorem, then (2.6) becomes

$$\iint_{\Sigma} (\text{curl } \mathbf{E}\cdot d\Sigma) = -K\frac{d}{dt}\iint_{\Sigma} (\mathbf{B}\cdot d\Sigma)$$

where the surface Σ over which the integral on the left is taken also encloses the circuit C_0.

When the surface Σ is fixed with respect to the observer, the time derivative operation on the right hand side equals the partial derivative of $\mathbf{B}$ with respect to time, and so can be taken within the integral sign. Also the circuit C_0 and the surface Σ are completely arbitrary. Hence

$$\text{curl } \mathbf{E} = -K\frac{\partial \mathbf{B}}{\partial t}$$

This is the differential form of the law.

If magnetic induction $\mathbf{B}$ is expressed in electromagnetic units and electric intensity $\mathbf{E}$ in electrostatic units then such a system is called the Gaussian or mixed system of units. Experiments have shown that in this system K equals $1/c$ where c is the velocity of light in vacuum, and so approximately $c = 3 \times 10^{10}$ cm/s. Hence

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.7}$$

Another system of units, also often used in literature, is the MKS system, or the practical and rationalized or Giorgi units, where M is metre, K kilogram and S seconds.

4. It is found that the magnetomotive force $\left(= \oint_{C_0} (\mathbf{H}\cdot d\mathbf{l})\right)$ in the closed circuit C_0 is linearly proportional to the volume current I and to the rate of change in time of the flux of $\mathbf{D}$, that is,

$$\oint_{C_0} (\mathbf{H}\cdot d\mathbf{l}) = \alpha I + \beta\frac{d}{dt}\iint_{\Sigma} (\mathbf{D}\cdot d\Sigma)$$

where α and β are the constants of proportionality. Experiments have shown that if I is in statampères and $\mathbf{D}$ is in statvolts/cm, that is, in electrostatic units, then $\alpha = 4\pi/c$, $\beta = 1/c$. Hence the integral form of a medium at rest is

$$\oint_{C_0} (\mathbf{H}\cdot d\mathbf{l}) = \frac{4\pi}{c} I + \frac{1}{c} \iint_{\Sigma} \left(\frac{\partial}{\partial t} \mathbf{D}\cdot d\Sigma \right) \tag{2.8}$$

Again applying Stoke's theorem on the integral in the left hand side and arguing as earlier, we get the differential form of the law:

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \tag{2.9}$$

where $I = \iint_{\Sigma} (\mathbf{j}\cdot d\Sigma)$; **j** is the current vector crossing unit area of Σ at the point of integration*. The quantity $\partial\mathbf{D}/\partial t$ has dimensions of current density. It was called displacement current by Maxwell because it was due to displacement of dipole charges in dielectrics due to the application of an electric field. The quantity **j** is called conduction current because it is related to the motion of free charges. In dielectrics the conduction current is ignored, and in conductors the displacement current.

3. Lorentz force

The force density **F**, depending on the field of electric charges and currents in unit volume at the field point *P*, is called the Lorentz force and can be obtained experimentally:

$$\mathbf{F} = \rho\mathbf{E} + \frac{1}{c} [\mathbf{j} \times \mathbf{B}] \tag{3.1}$$

where ρ is the density of total charges, **E** is the electric field, **j** is the current density and **B** is the magnetic induction at *P*. This force is the cause of motion of charges if they are set free in the field of **E** and **B**. Taking **v** to be the macroscopic velocity of motion of the average charge density ρ at *P*, we have

$$\mathbf{j} = \rho\mathbf{v},\ \mathbf{F} = \rho\mathbf{E} + \frac{\rho}{c} [\mathbf{v} \times \mathbf{B}] \tag{3.2}$$

When the charges are forced to be at rest, $\mathbf{v} = 0$, and so

$$\mathbf{F} = \rho\mathbf{E}$$

*Electric currents are moving electric charges. The current I is defined as the rate at which a charge is transported through a conductor. Hence $I = dq/dt$, or, $I = Anqv$ where q is the charge, t is the time, A is the cross sectional area of the conductor, n is the number of charges per unit volume and v is the velocity of their motion. The flow rate of charges per unit area is known as the current density j. Hence $jA = I$ and so $j = nev$. When the current I is distributed uniformly over the cross section of the conductor the current density j may be treated as a scalar quantity. But if the current distribution is not uniform over the cross section, the current density is the average current density over the cross section. At a point on the plane area δA, perpendicular to the direction of the flow, therefore, if the current element vector it is $\delta\mathbf{I}$, the current density **j** is defined as $\mathbf{j} = \lim_{\delta A \to 0} \delta\mathbf{I}/\delta A$.