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ABSTRACT

We study generalized Dyson series and their representation by generalized Feynman diagrams as well as the closely related topic of Feynman's time-ordered operational calculus for noncommuting operators. These perturbation series are obtained by replacing ordinary Lebesgue measure in the time integration involved in the Feynman-Kac functional by an arbitrary Lebesgue-Stieltjes measure; we then calculate the Wiener and Feynman path integrals of the corresponding functional. Our Dyson series provide a means of carrying out the "disentangling" which is a crucial element of Feynman's operational calculus. We are also able to treat far more general functionals than the traditional exponential functional; in fact, the class of functionals dealt with forms a rather large commutative Banach algebra.

An intriguing aspect of the present theory is that it builds bridges between several areas of mathematical physics, operator theory and path integration. Combinatorial considerations permeate all facets of this work.

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CONTENTS

LIST OF FIGURES

0. INTRODUCTION AND PRELIMINARIES

0.1. Introduction.

Let $C = C[0,t]$ denote the space of continuous functions on $[0,t]$ with values in $\mathbb{R}^N$. In the study of the Feynman-Kac formula and of the Feynman integral, a particular class of functionals on $C[0,t]$ has been of paramount importance:

$$F(y) := \exp\{\int_{(0,t)} \theta(s,y(s))ds\}, \tag{0.1}$$

where the "potential" θ is a complex-valued function on $[0,t] \times \mathbb{R}^N$. In this paper, we consider analytic functions $f(z)$ of the functional

$$F_1(y) := \int_{(0,t)} \theta(s,y(s))d\eta(s), \tag{0.2}$$

where η belongs to $M = M(0,t)$, the space of complex Borel measures on $(0,t)$ [3, Chap. 4; 42, pp. 19-23]. We calculate the associated Wiener integral and, after analytic continuation, obtain the corresponding Feynman integral. In carrying out the Wiener path integral, it is advantageous to use the unique decomposition of the measure η, $\eta = \mu + \nu$, into its continuous part μ and its discrete part ν [3, p. 12; 42, p. 22]. This decomposition, with appropriate care taken with the time-ordering and the combinatorics involved, leads to a "generalized Dyson series". If $f(z) = \exp(z)$ and $\eta = \mu =: \ell$, where ℓ is ordinary Lebesgue measure on $(0,t)$, the perturbation series, in "real time", is just the classical Dyson series [8; 45, Chap. 11.f].

The additional flexibility provided by the use of Lebesgue-Stieltjes measures in this context has many implications, allowing us to broaden and unify known concepts and to introduce new ones having an interest in their own right. When $\eta = \mu$ is a continuous measure, the generalized Dyson series has the same formal appearance as in the classical case. However, even when μ is absolutely continuous, very different interpretations are suggested; for example, all the mass could be concentrated

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near a single instant τ. Moreover, μ could have a nontrivial singular part.

When η has a nonzero discrete part, the form of the generalized Dyson series changes markedly and genuinely new phenomena occur. The combinatorial structure of the series is much more complicated even when ν is finitely supported. For instance, additional summations appear as well as powers of the potential θ evaluated at fixed times. Indeed, some of the combinatorial complications and nearly all of the analytic difficulties are found in the simple case $\eta = \mu + \omega\delta_\tau$, where δ_τ is the Dirac measure at τ. Accordingly, we discuss this prototypical example in detail in Section 1 and elsewhere in the paper and use it as a conceptual aid to the general development.

Another particular case of interest is obtained when $\eta = \nu$ is a purely discrete measure with finite support:

$$\nu = \sum_{p=1}^{h} \omega_p \delta_{\tau_p}, \text{ with } 0 < \tau_1 < \dots < \tau_h < t. \tag{0.3}$$

By considering the exponential functional [i.e., by letting $f(z) = \exp(z)$] and further specializing, we will see in Example 3.3 that the series reduces to a single term, the familiar h-th Trotter product. Now approximating Lebesgue measure ℓ by discrete measures of the form (0.3) and applying a stability theorem with respect to the measures (Theorem 4.3), we establish connections with the Trotter product formula [2,28,40,48,...].

One can also use the stability theorem to see the relationship between the relatively simple perturbation series corresponding to continuous η and more complex Dyson series. We make this explicit in Example 4.2 where δ_τ is approximated by absolutely continuous measures whose densities are given by a δ-sequence.

Our generalized Dyson series can be represented graphically by generalized Feynman diagrams. The n-th term of the classical Dyson series corresponds to a single connected Feynman graph. Here, however, the n-th term of the generalized Dyson series gives rise to many disconnected components, one for each summand. The complex combinatorial structure of the generalized Dyson series is accurately reflected in the generalized Feynman graphs, and the reader may find it helpful, after a brief look at Section 5, to draw such graphs while following the proofs and examples in Sections 1-4.

Different situations may lead to generalized Dyson series and Feynman graphs with diverse combinatorial structures as is illustrated in Section 3 where we present miscellaneous examples. We remark that, up to this point, we have discussed only cases involving a single measure η and a single potential θ. However, we can, for example, treat functionals formed by composing an analytic function of several complex variables with functions of the form (0.2). We mention in particular Example 3.6 which we use to make explicit some of the ties with Feynman's operational calculus.

Feynman's time-ordered operational calculus, introduced in [11], is based on the interesting observation that noncommuting operators A and B can be treated as though they commuted; a time index is attached to them to indicate the order of operation. More specifically, Feynman suggests writing

$$(0.4) \qquad A(s_1)B(s_2) = \begin{cases} BA \text{ if } s_1 < s_2 \\ AB \text{ if } s_2 < s_1 \\ \text{undefined if } s_1 = s_2. \end{cases}$$

One then performs the desired calculations just as if A and B were commuting. Eventually one wants to restore the conventional ordering of the operators; Feynman refers to this as "disentangling". He says [11, p.110]: "The process is not always easy to perform and, in fact, is the central problem of this operator calculus".

Our generalized Dyson series provide a means of carrying out this disentangling process for a rather large class of operators. It is the use of path integration that enables us to accomplish this. Some possible relations with path integration were already suggested in Feynman's paper [11, p. 108 and Appendices A-C, pp. 124-127] and in the book of Feynman and Hibbs [12, pp. 355-356].

We note that we will, for example, be integrating expressions similar to the left-hand side of (0.4) over a square $(0,t) \times (0,t)$, and, when this is done with respect to measures with nonzero discrete part, one cannot ignore the diagonal of the square. We point out that Feynman's convention (0.4) is suited for Lebesgue measure ℓ, a continuous measure, so that the diagonal $s_1 = s_2$ of the square is a set of measure zero. In this sense, our theory is broader than parts of Feynman's operational calculus.

We are using the expression "Feynman's operational calculus" as though it has a precise meaning. However, a key problem is to give a precise definition and interpretation of this calculus and to demonstrate how to use it effectively in particular in carrying out the disentangling process and in developing a functional calculus. The reader might be interested and surprised to read Feynman's own comments [11, p. 108] on the difficulty of putting his methods on a rigorous basis and on the need for further mathematical development.

The class of functionals on Wiener space that we are able to treat is quite large. In fact, under pointwise mulitplication and equipped with a natural norm, it forms a commutative Banach algebra A consisting of certain series of products of functionals of the form (0.2). With the help of the basic results of Section 2, we show in Section 6 that each functional in A possesses operator-valued Wiener and Feynman integrals, enlarging in the process the class of functionals for which the operator-valued Feynman integral is known to exist. Further, each of these operators can be disentangled in the form of a generalized Dyson series.

Related but much smaller Banach algebras of functionals were studied by Johnson and Skoug in [18 and 19]; [19, pp. 121-123] is especially relevant. The functionals in [19] are generated by functionals of the form (0.2) with θ varying but with η fixed as Lebesgue measure. The resulting Dyson series are much simpler. The emphasis in [19] was somewhat different, and, in particular, no attempt was made to relate the results to Feynman's operational calculus.

Feynman's paper [11], in conjunction with the present work and that of Lapidus in [33,34], suggests additional questions which we anticipate investigating in a subsequent paper that will further develop Feynman's operational calculus for noncommuting operators.

We mention the works of Nelson [41] and Maslov [36] which are also related to Feynman's operational calculus. They have little in common, and both are very different in spirit from the present paper. In particular, the connections with path integration as well as the complicated combinatorics associated with the disentanglement that leads to our generalized Dyson series do not appear in either of [41] or [36].

It is reasonable to refer to the functional

$$F(x) := \exp\left(\int_{(0,t)} \theta(s,x(s))\,d\eta(s)\right) \tag{0.5}$$

as the Feynman-Kac functional with Lebesgue-Stieltjes measure η. It is natural to ask if the corresponding operator, considered as a function of time, satisfies a differential equation analogous to the heat or Schrödinger equations. This is the case, as is shown by Lapidus in [33, 34] where a "Feynman-Kac formula with a Lebesgue-Stieltjes measure" is established and related results are given. (See Kac's papers [23; 24, pp. 62-65] for the classical Feynman-Kac formula.) For an exponential functional of the type (0.5), for instance, the study conducted in [33, 34] reveals the distinct roles played by the continuous part and the discrete part of η. It also makes explicit connections with the theory of the product integral [5].

We now describe briefly the organization of this paper. In the remainder of the present section, we introduce notation and give two preliminary lemmas.

In Section 1, we discuss the prototypical example $\eta = \mu + \omega\delta_\tau$ mentioned above; our most detailed analytic proofs are given in this case.

Generalized Dyson series for the full class of functionals treated in this paper are obtained in Section 2. Some readers might wish to consult this section only briefly on a first reading.

Section 3 may be particularly helpful to the reader as it deals with a variety of concrete examples of perturbation expansions. The emphasis in Sections 2 and 3 is largely on the combinatorics.

In Section 4, we give theorems insuring stability with respect to the potentials and with respect to the measures. We also give some applications of the stability theorems for measures.

We present, in Section 5, a graphical representation of our generalized Dyson series in terms of generalized Feynman diagrams.

In Section 6, we show that the general class of functionals treated in Section 2 forms a commutative Banach algebra, and we discuss the related functional calculus. We finish with a discussion of some connections with Feynman's operational calculus.

Possible physical interpretations are provided in various places throughout the paper.

A great variety of Feynman diagrams and perturbation expansions appear in the physics literature. We should make it clear that we do not claim here to be generalizing all of those.

Parts of the present paper were announced in [17].

0.2. Notation and Preliminaries.

In A through I below, we recall some facts and introduce most of the notation which we will require. With the possible exception of G and I, we suggest that the reader go over the material quickly and then return to it if and when it is necessary.

First we mention some general references: For the theory of the Wiener process and applications of path integration, the reader may wish to consult [13,14,24,25,46,50]. For semigroup theory, we mention [6, Chap. 8; 15; 26]; for the theory of the Bochner integral, we refer to the treatise of Hille and Phillips [15, Chap. III]. Finally, the basic facts of measure theory used in this paper can be found in [42, §§1.3 and 1.4, pp. 12-26] and [3,43,49].

A. $\mathbb{C}$, $\mathbb{C}_+$, $\tilde{\mathbb{C}}_+$: These denote, respectively, the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part.

B. $L^2(\mathbb{R}^N)$: The space of Borel measurable, $\mathbb{C}$-valued functions ψ on $\mathbb{R}^N$ such that $|\psi|^2$ is integrable with respect to Lebesgue measure on $\mathbb{R}^N$.

C. $L^\infty(\mathbb{R}^N)$: The space of Borel measurable, $\mathbb{C}$-valued functions on $\mathbb{R}^N$ which are essentially bounded.

More formally, the elements of $L^2(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ are equivalence classes of functions, with ψ_1 and ψ_2 said to be equivalent if they are equal almost everywhere (a.e.) with respect to Lebesgue measure.

D. $\mathcal{L}(L^2(\mathbf{R}^N))$: The space of bounded linear operators from $L^2(\mathbf{R}^N)$ into itself.

The notation $\|\cdot\|$ will be used both for the norm of vectors and for the norm of operators; the meaning will be clear from the context.

E. The semigroup $\exp(-zH_0)$: We give some facts which we will use frequently concerning the holomorphic semigroup $\{\exp(-zH_0)\}_{z\in \tilde{C}_+}$ generated by the "free Hamiltonian" $H_0 = -(1/2)\Sigma_{\alpha=1}^N \partial^2/\partial x_\alpha^2$ in $L^2(\mathbf{R}^N)$. (See [26, Chap. IX, §1.8, pp. 495-497].) We use notation convenient for our purposes. The operators $\{\exp[-s(H_0/\lambda)]: s > 0, \lambda \in \tilde{C}_+\}$ are all in $\mathcal{L}(L^2(\mathbf{R}^N))$ and satisfy:

$$\|\exp[-s(H_0/\lambda)]\| \leq 1. \tag{0.6}$$

In fact, when $\lambda \in \tilde{C}_+$ is purely imaginary, $\exp[-s(H_0/\lambda)]$ is a unitary operator. As a function of λ, $\exp[-s(H_0/\lambda)]$, also denoted by $e^{-s(H_0/\lambda)}$ in this paper, is analytic in C_+ and continuous in the strong operator topology (or strongly continuous) in $\tilde{C}_+$. (Recall that for operator-valued (or for vector-valued) functions, all the natural notions of analyticity coincide. See [15, §3.10, esp. Theorem 3.10.1, p. 93].) Next, we state a familiar explicit formula for the operator $\exp[-s(H_0/\lambda)]$. Given $\psi \in L^2(\mathbf{R}^N)$,

$$(\exp[-s(H_0/\lambda)]\psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right)^{N/2} \int_{\mathbf{R}^N} \psi(u)\, \exp\left[\frac{-\lambda\|u-\xi\|^2}{2s}\right] du. \tag{0.7}$$

The integral in (0.7) exists as an ordinary Lebesgue integral for $\lambda \in C_+$, but, when λ is purely imaginary and ψ is not integrable, the integral should be interpreted in the mean just as in the theory of the Fourier-Plancherel transform.

As is well known, the (negative) normalized Laplacian H_0 is the generator of the Brownian motion on $\mathbf{R}^N$: it follows, in particular, that the semigroup $\{\exp(-sH_0)\}_{s>0}$ is intimately connected with Wiener measure m defined in I below. (See [13, Chap. 3; 14, Chap. 3; and esp. 46, Chap. II].)

F. $M(0,t)$: Let $t > 0$ be fixed. $M(0,t)$ will denote the space of complex Borel measures η on the open interval $(0,t)$. For information on such spaces of measures, see, for example, [3, Chap. 4]. Given a Borel subset B of (0,t), the total variation measure $|\eta|$ is defined by $|\eta|(B) = \sup\{\Sigma_{j=1}^{n}|\eta(B_j)|\}$, where the supremum is taken over all finite partitions of B by Borel sets (see [3, p. 126]). M(0,t) is a Banach space under the natural operations and the norm

$$\|\eta\| := |\eta|(0,t). \tag{0.8}$$

A measure μ in M(0,t) is said to be <u>continuous</u> if $\mu(\{\tau\}) = 0$ for every τ in (0,t). In contrast, ν in M(0,t) is <u>discrete</u> (or is a "pure point measure" in the terminology of Reed and Simon [42]) if and only if there is an at most countable subset $\{\tau_p\}$ of (0,t) and a summable sequence $\{\omega_p\}$ from C such that

$$\nu = \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}, \tag{0.9}$$

where δ_{τ_p} is the Dirac measure with total mass one concentrated at τ_p [3, p. 12]. Every measure $\eta \in M(0,t)$ has a unique decomposition, $\eta = \mu + \nu$, into a continuous part μ and a discrete part ν [42, Theorem I.13, p. 22]. We will make frequent use of such decompositions.

We work with the space M(0,t) throughout, but M[0,t] could be treated without any essential complications. However, allowing η to have non-zero mass at 0 introduces additional alternatives which we have chosen to avoid.

G. $L_{\infty 1;\eta}$: Let $\eta \in M(0,t)$. A C-valued, Borel measurable function θ on $(0,t) \times \mathbb{R}^N$ is said to belong to $L_{\infty 1;\eta}$ if

$$\|\theta\|_{\infty 1;\eta} := \int_{(0,t)} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s) < +\infty. \tag{0.10}$$

Note that if $\theta \in L_{\infty 1;\eta}$, then $\theta(s,\cdot)$ must be in $L^\infty(\mathbb{R}^N)$ for η-a.e. s in (0,t). If one makes the usual identification of functions which are equal $\eta \times$ Lebesgue-a.e., the mixed norm space $L_{\infty 1;\eta}$, equipped with the norm $\|\cdot\|_{\infty 1;\eta}$, becomes a Banach space. Note that all bounded, everywhere defined, Borel measurable functions on $(0,t) \times \mathbb{R}^N$ are in $L_{\infty 1;\eta}$ for every η in M(0,t).

The reader will see further on that the norm (0.10) appears in our estimates in a natural way.

The functions θ will be interpreted physically as potentials. The condition that θ be in $L_{\infty 1;\eta}$ is rather minimal in most respects. No smoothness is required, and θ is allowed to be time-dependent and $\mathbb{C}$-valued. The use of $\mathbb{C}$-valued functions θ will enable us, in particular, to treat the diffusion case (or "imaginary time" case) as well as the quantum mechanical case (or "real time" case). (See Remark 0.3 below.) The importance of $\mathbb{C}$-valued potentials in the study of decay systems in quantum mechanics is discussed thoroughly in the recent book of Exner [9]. Certainly, the most serious restriction in our assumptions is that $\theta(s,\cdot)$ be essentially bounded for η-a.e. s. However, even this condition seems quite reasonable in light of our goal of obtaining rigorously justified perturbation series valid in the quantum mechanical case.

If $\theta \in L_{\infty 1;\eta}$ and if $\eta = \mu + \nu$ is decomposed into its continuous and discrete parts, then it is not difficult to show that $\theta \in L_{\infty 1;\mu} \cap L_{\infty 1;\nu}$ and

(0.11) $$||\theta||_{\infty 1;\eta} = ||\theta||_{\infty 1;\mu} + ||\theta||_{\infty 1;\nu} .$$

H. <u>The multiplication operators</u> $\theta(s)$: We remind the reader that the operator of multiplication by a function in $L^\infty(\mathbb{R}^N)$ belongs to $\mathcal{L}(L^2(\mathbb{R}^N))$ and has operator norm equal to the essential supremum of the function. (See, e.g., [26, Example 2.11, p. 146].) For us, the L^∞-functions that arise will be of the form $\theta(s,\cdot)$, where $\theta \in L_{\infty 1;\eta}$. It will be convenient to let $\theta(s)$ denote the operator of multiplication by $\theta(s,\cdot)$, acting in $L^2(\mathbb{R}^N)$. The operator norm $\|\theta(s)\|$ then satisfies

(0.12) $$\|\theta(s)\| = \|\theta(s,\cdot)\|_\infty .$$

I. <u>The operator-valued function space integrals</u> $K_\lambda(F)$, $\lambda \in \tilde{\mathbb{C}}_+$: First, let $C_0 = C_0[0,t]$ be the space of $\mathbb{R}^N$-valued continuous functions x on [0,t] such that x(0) = 0. We consider C_0 as equipped with N-dimensional Wiener measure m which is just the product of N one-dimensional Wiener measures [14,46,50]; recall that m is a probability measure on C_0 [50, Chap. 7].

DEFINITION 0.1. Let F be a function from $C[0,t]$ to $\mathbb{C}$. Given $\lambda > 0$, $\psi \in L^2(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$, we consider the expression

$$(0.13) \qquad (K_\lambda(F)\psi)(\xi) = \int_{C_0} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(t)+\xi)dm(x).$$

The operator-valued function space integral $K_\lambda(F)$ exists for $\lambda > 0$ if (0.13) defines $K_\lambda(F)$ as an element of $\mathcal{L}(L^2(\mathbb{R}^N))$. If, in addition, $K_\lambda(F)$, as a function of λ, has an extension (necessarily unique) to an analytic function on $\mathbb{C}_+$ and a strongly continuous function on $\tilde{\mathbb{C}}_+$, we say that $K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$. When λ is purely imaginary, $K_\lambda(F)$ is called the (analytic) operator-valued Feynman integral of F.

REMARK 0.1. The function F in Definition 0.1 (often referred to as a "functional" in the physics literature), need not be everywhere defined; however, in order to have $K_\lambda(F)$ defined for all $\lambda > 0$, it must be the case that, for every $\lambda > 0$, $F(\lambda^{-1/2}x+\xi)$ is defined for $m \times$ Lebesgue-a.e. $(x,\xi) \in C_0 \times \mathbb{R}^N$.

Given another function G on $C[0,t]$, we say that F is equivalent to G ($F \sim G$) if, for every $\lambda > 0$, $F(\lambda^{-1/2}x+\xi) = G(\lambda^{-1/2}x+\xi)$ for $m \times$ Lebesgue-a.e. $(x,\xi) \in C_0 \times \mathbb{R}^N$. [Note that if $F \sim G$ and $K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$, then $K_\lambda(G)$ exists and $K_\lambda(F) = K_\lambda(G)$ for $\lambda \in \tilde{\mathbb{C}}_+$.] This equivalence, which may appear strange to begin with, is necessitated by the pathology of Wiener measure under scale change and the fact that infinitely many scale changes (corresponding to all $\lambda > 0$) are involved here. See [20] for a discussion of this and related matters.

Interest in the "Feynman integral" stems from Feynman's 1948 paper [10] which gave a formula for the evolution of a quantum system in terms of certain heuristically defined path integrals. Making Feynman's ideas mathematically rigorous in a useful way has proven difficult. There have been many approaches taken to this problem; that is, many Feynman integrals. A good introduction to this topic as well as many further references can be found in the recent book of Exner [9, esp. Chaps. 5 and 6]. For λ purely imaginary, $K_\lambda(F)$, as above, provides one way to make Feynman's definition precise.