Lecture Notes in Mathematics

Jan Stevens

Deformations of Singularities

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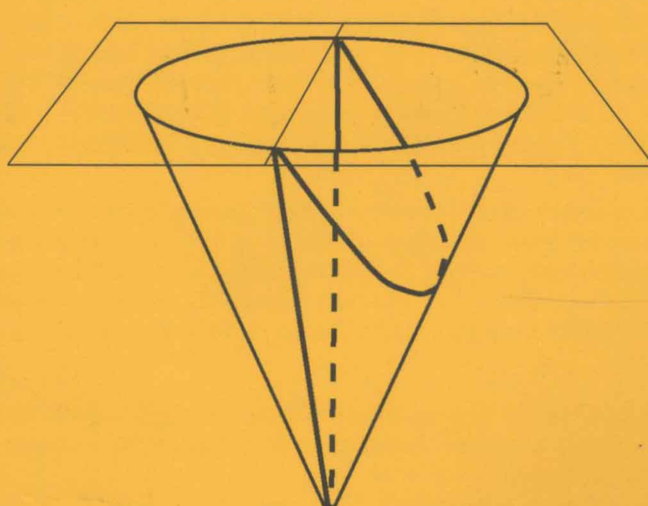

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Jan Stevens

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Author

Jan Stevens

Matematik
Göteborgs Universitet
Chalmers Tekniska Högskola
41296 Göteborg, Sweden

e-mail: stevens@math.chalmers.se

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Preface

Deformation theory has the reputation of being a difficult subject, for which no good literature is available. The main obstacle to understanding most of the existing texts is their level of generality. But many of the problems one has to confront already occur in the well established theories of deformations of compact complex manifolds (Kodaira–Spencer theory) and of universal unfoldings of function germs (Thom–Mather theory). When writing my Habilitationsschrift in Hamburg, of which these notes are an outgrowth, I decided to start with some introductory chapters on deformation theory. The warning to prospective authors of popular books on science, that each mathematical formula will cut down the readership by half, applies mutatis mutandis to the use of cofibred categories. They seem abstract nonsense, but in fact versality has its most natural formulation in these terms. There is a certain discongruity between general theory and practical computations (of which I have been doing quite a lot during the last years). One point I want to make, is that both can be understood as the problem of solving a deformation equation.

Having one's papers (since 1989) on file tempts one to 'recycle' old work. But I do hope that the slow process of revising, which led to the present text, has at least removed some of the mistakes.

It is a pleasure to thank all those who contributed in one form or another to the existence of these notes. Especially I want to thank Kurt Behnke for the many discussions during our joint time in Hamburg. For conversations, which among other things helped me shape my ideas, I am grateful to Duco van Straten, Ragnar Buchweitz, Jan Christophersen, Theo de Jong, Miles Reid and Jonny Wahl (this is a non-exhaustive list). I thank Oswald Riemenschneider and the participants of his 'Seminar über Komplexe Analysis' in Hamburg. I thank the former European Singularity Project and its successor for the environment it created, and its organisers for their efforts. I especially thank Gert-Martin Greuel, Dirk Siersma, and Terry Wall. Thanks to Klaus Altman for finding mistakes in the originally submitted Habilitationsschrift.

I could not have computed so many examples without computer use, more specifically the computer program *Macaulay* [BS]. Therefore thanks to Dave Bayer and Mike Stillman, and David Eisenbud for his scripts. I managed to

enlighten the text with some real pictures thanks to the program *surf* by Stephan Endraß [En].

Göteborg, August 2002.

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Addresses:

Professor J.-M. Morel, CMLA,
Ecole Normale Supérieure de Cachan,
61 Avenue du Président Wilson, 94235 Cachan Cedex, France
E-mail: Jean-Michel.Morel@cmla.ens-cachan.fr

Professor F. Takens, Mathematisch Instituut,
Rijksuniversiteit Groningen, Postbus 800,
9700 AV Groningen, The Netherlands
E-mail: F.Takens@math.rug.nl

Professor B. Teissier, Université Paris 7
Institut Mathématique de Jussieu, UMR 7586 du CNRS
Equipe "Géométrie et Dynamique", 175 rue du Chevaleret
75013 Paris, France
E-mail: teissier@math.jussieu.fr

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Table of Contents

Introduction

Deformation theory has its origins in the theory of moduli; '[eine Theorie], die für uns, als wir begannen, lange Zeit mit einem Schleier umhüllt war. Riemann sagt, daß bei beliebiger birationaler Transformation des Gebildes nicht nur die Zahl p ungeändert bleibt, sondern (für $p > 1$) auch noch $3p - 3$ Konstanten, die er „*Moduln*" des Gebildes nennt. Diese Moduln sind einfach die absoluten Invarianten, welche die Normalkurve C_{2p-2} gegenüber linearer Transformationen ihrer homogenen Koordinaten aufweist! ... Es ist doch die lineare Invariantentheorie, die die Probleme beherrscht, aber nur, wenn man sie richtig in Ansatz bringt!' [Kle, p. 310]. The situation for Riemann surfaces is the following: for the topological classification one discrete invariant suffices, the geometric genus p_g, but the analytical structure depends on continuous parameters, called moduli, which one would like to consider as coordinates on a reasonably nice space, whose points are in one-to-one correspondence with the isomorphism classes of Riemann surfaces. KLEIN indicates a way to construct the moduli space as orbit space: it is the quotient of the finite dimensional space parametrising canonically embedded curves (this almost works; to include the hyperelliptic curves one has to use the tri-canonical embedding instead). For other classification problems one can try to do the same.

The fundamental discovery of KODAIRA and SPENCER is that already for surfaces moduli spaces in general do not exist [KS]. Instead one has to settle for 'local moduli', as it is sometimes called; in the terminology of these notes, it is the semi-universal deformation of a given object. We start out with a given surface and deform the analytical structure. Now we want a space which not only parametrises all isomorphism classes of nearby structures, but also all germs of continuous families, which contain the given surface as special element. Such a space exists and the 'number of moduli' for the given surface can be defined as the dimension of its tangent space. Nearby surfaces can depend on less moduli. This forces the 'moduli space' to be non-Hausdorff, which means that it cannot exist, at least as the reasonably nice space we wanted. Such exceptional behaviour is the rule in the Thom–Mather theory of unfoldings of functions (cf. [AGV]). Here finite determinacy allows the reduction of the classification problem to a finite dimensional one: the Lie group of k-jets of coordinate transformations acts on k-jets of functions, and the semi-universal unfolding of a function is obtained from a transversal slice to the orbit of its jet under this action. For many deformation problems the picture is in principle the same (cf. [Bi2]), but one has to work with infinite dimensional spaces, which lead to enormous analytical problems.

A second source of deformation-like problems in algebraic geometry is the theory of (linear) systems of curves on surfaces and its generalisations, see [Mu2]. The Zariski tangent space to the family is $H^0(C, N_C)$, where N_C is the normal bundle, and obstructions lie in $H^1(C, N_C)$; the question, if the family in question is smooth of dimension $h^0(C, N_C)$, is classically known as the problem of the completeness of the characteristic linear system. The functorial language to treat such problems has been developed by GROTHENDIECK [Gro]. He also realised that his formalism is the correct one for deformation theory in general and that of compact complex manifolds in particular.

The first half of these notes treats the general theory of deformations and of deformations of singularities in particular. As illustration some quite specific computations are given. The remaining chapters consider more specific problems, specially on curves and surfaces.

Deformations of singularities (i.e., germs of analytic spaces) can be described in terms of deformations of the local ring, in the sense of [Ge]. The underlying vector space is not changed, but the multiplication map is perturbed. That the deformed multiplication is again associative is a highly non-trivial condition. Since the work of SCHLESSINGER (cf. [Art3, Schl2]) and TYURINA [Ty] we have a direct definition in terms of defining equations, see Chap. 1. The existence of versal deformations for isolated singularities has been shown by GRAUERT in a much cited (but not read?) paper [Gra]. Using a power series Ansatz, computations are possible. This is all what can be said in this generality; starting with a system of equations $(f_1(x), \ldots, f_k(x))$, describing the singularity, one finds a more complicated system $(F_1(x,t), \ldots, F_k(x,t))$ (where $F_i(x,0) = f_i(x)$), and in general also equations $g_j(t)$ between the deformation parameters.

'*Grauert* löste die zentralen Probleme der Deformationstheorie mit so schlagender Gewalt, daß der ganzen Theorie darüber fast der Atem ausging.' [Remmert, Laudatio at the presentation of the von Staudt price to Grauert, see DMV-Mitteilungen 1/93]. The existence of versal deformations should really be the beginning of the theory. The situation however is that the existence of a formally versal, formal object is often easy, and the result states that an analytically versal, analytic object exists. Hard analysis is needed for the difficult proof, but it sheds no further light on the objects, and in particular it gives no practical way to compute. We give in Chap. 6 BUCHWEITZ' example of a smooth affine (elliptic) curve, which is formally rigid, but has non-trivial analytic deformations. In some sense, this is a trivial example, and it would be interesting to have a similar example with a deformation problem 'in real life'; of course, in a 'nice world' it does not exist.

It is useful to formulate a deformation problem in terms of a more or less explicit deformation equation; for compact complex manifolds this is achieved by the integrability condition for almost complex structures, see e.g. [Ku]:

$$\bar{\partial}\vartheta + \tfrac{1}{2}[\vartheta, \vartheta] = 0.$$

The most practical way to obtain concrete solutions, is to use a power series Ansatz. As one wants an answer in finite time, one really looks for polynomial solutions. The equation can also be used to obtain existence results, say by using some form of the implicit function theorem. For deformations of singularities one can get the same type of deformation equation from the general theory of the cotangent complex. In concrete computations it boils down to the formalism of lifting relations.

Knowing the existence of versal deformations of singularities we can go on and ask questions about the structure of the base space (is it reduced, what is the number of components?), or ask if for some t_0 the fibre $(F_1(x,t_0), \dots, F_k(x,t_0))$ is smooth. In general these question cannot be answered, because the equations are just one enormous mess. However, for specific, well-chosen examples things look better; in most cases the distinguishing factor is some extra symmetry on the equations, which extend through the whole computation. In hand calculations the symmetry can be used to organise them; on a machine the symmetry enters mostly indirectly, in that only certain monomials can occur in an expression, which is therefore smaller and easier to understand.

A striking example of the importance of visible symmetry, is the contrast between the ease, with which in Chap. 3 the versal deformation of L^n_n (the curve, consisting of the coordinate axes in $\mathbb{C}^n$) is computed, and long computations in Chap. 11 for the isomorphic curve in determinantal form (only for $n = 4$; these computations are a prelude to those for the monomial curve (t^4, t^5, t^6, t^7), which is in a natural way determinantal).

It is desirable to have a systematic way to write down equations. For the famous example of the cone over the rational normal curve of degree four, there are two determinantal representations, and in fact they lead to the two components of the base space. More examples are given in Chap. 12 on *formats*. Presumably, there is no general statement, and we are spoilt by the simple examples that are found first.

Explaining smoothing components (of rational surface singularities) is the goal of KOLLÁR's conjectures [Kol]. Originally it was thought that (if true) they would give a method to find the smoothing components, but apart from the case of quotient singularities [KSh], it seems that one has first the components, and then the corresponding P-modifications. The pessimistic view is that one impossible problem is replaced by another. The interest of the conjectures is the new understanding, to which they lead. In Chap. 14 I give a number of examples of non rational singularities with P-modifications, explaining the components, and extend the conjectures to all surface singularities.

For curve singularities no interpretation of smoothing components is known. In this case all smoothing components have the same dimension. Chap. 13 contains the first example of a curve with several smoothing components: it is L^6_{14}, 14 lines in general position through the origin in $\mathbb{C}^6$. I originally performed the calculation to decide whether the singularity was smoothable at all. The case of L^6_{14} was left open in my earlier work on smoothability of certain cones over points [St1].

Cones over curves form the subject of the last two chapters. Powerful methods exist to compute T^1 for surface singularities, without using explicit equations. For cones over curves the bundle of principal parts comes in, and with it WAHL's Gaussian map (cf. [Wa6]). The computation of $T^1(-1)$ is the most difficult; much of the work on the Gaussian map is connected with vanishing results for this case. The most complete results on interesting deformations are obtained by Sonny TENDIAN [Te1]. With a trick, which basically is contained in [Mu3], one sees that for non hyperelliptic curves, embedded with a non-special line bundle L, the dimension of $T^1(-1)$ of the cone equals $h^0(C, N_K \otimes L^{-1})$, where N_K is the normal bundle of C in its canonical embedding. For low genus this gives quite precise information, because then the normal bundle N_K is easy to describe.

If S is a surface with C as hyperplane section, then one can degenerate S to the projective cone over C, or from another point of view, deform the projective cone over C to S; PINKHAM calls this construction 'sweeping out the cone' [Pin1]. Surfaces with hyperelliptic hyperplane sections were already classified by CASTELNUOVO, and the supernormal surfaces among them have degree $4g + 4$ [Cas]. They are rational ruled surfaces, and such surfaces come in two deformation types; therefore there are at least two smoothing components. A computer computation of the versal deformation in negative degree with *Macaulay* [BS] gave for an example with $g = 2$ the number of 32 smoothing components. I show that cones over hyperelliptic curves of degree $4g + 4$ have 2^{2g+1} smoothing components (the case $g = 3$ is exceptional).

1 Deformations of singularities

The definition of a deformation involves the notion of flatness, which accounts for the difficulties in explaining and understanding it. In the mid sixties MUMFORD wrote: "The concept of flatness is a riddle that comes out of algebra, but which is technically the answer to many prayers." [Mu1, p. 295]. Intuitively, in a flat family the fibres depend continuously on the points of the parametrising base space. We assume that the reader is familiar with flat morphisms [Ha, III.9], [Fi, 3.11]; the purpose of this section is to show the relevance for deformation theory. After some examples we eventually define a deformation of a space (germ) X_0 as a flat map $\pi: X \to S$ with $X_0 \cong \pi^{-1}(0)$. The term *deformation* is a convenient way of speaking, which emphasises the special role of X_0; if we consider the general fibre X_s, $s \neq 0$, as primary object, we speak of a *degeneration*, or *specialisation*.

Example. Consider a quartic curve C_0 in $\mathbb{P}^3$ with a double point. To be specific, let C_0 be the image of the map $f: \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(x_0 : x_1 : x_2 : x_3) = (s_1^4 - s_2^4 : s_1^3 s_2 : s_1^2 s_2^2 : s_1 s_2^3) \, .$$

The curve C_0 is a complete intersection with equations

$$x_2^2 - x_1 x_3 \, , \qquad x_1^2 - x_0 x_2 - x_3^2 \, .$$

The double point lies in $(1:0:0:0)$.

By perturbing the map f to f_t we obtain in general a smooth rational quartic curve C_t. To see what happens in a neighbourhood of the double point, we take local coordinates (x, y, z), and equations $y = 0$, $xz = 0$. The curve germ $(C_0, 0)$ is the image of the multigerm $f: \mathbb{C} \cup \mathbb{C} \to \mathbb{C}^3$, given by $f(s_1) = (s_1, 0, 0)$ and $f(s_2) = (0, 0, s_2)$. Now consider the map $F: (\mathbb{C} \cup \mathbb{C}) \times \mathbb{C} \to \mathbb{C}^3 \times \mathbb{C}$, defined as $F(s_1, t) = (s_1, 0, 0, t)$ and $F(s_2, t) = (0, t, s_2, t)$. For $t \neq 0$ we have two skew lines in $\mathbb{C}^3 \times \{t\}$, which can be described by four equations:

$$yx = 0 \, , \qquad zx = 0 \, , \qquad y(y - t) = 0 \, , \qquad z(y - t) = 0 \, .$$

If we put $t = 0$ in these equations, we do not get the equations $y = 0$, $xz = 0$ of the image of f, but four equations $yx = zx = y^2 = zy = 0$; they describe