组合数学丛书

Zhe-Xian Wan

Design Theory

设计理论

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Preface

The present book is based on the lecture notes of a graduate course Design Theory which was given at the Center for Combinatorics of Nankai Unversity in spring of 2001. The lecture notes were scattered over the experts and students, modified year by year following some of their suggestions, and finally came to the present form.

The course consists of mainly the basic classical subjects of design theory, namely, balanced incomplete block designs, latin squares, t-designs and partially balanced incomplete block designs, and ends with association schemes.

The fundamental concepts of balanced incomplete block designs are given in Chapter 1 and various classical constructions appear in Chapters 2 and 3. Orthogonal latin squares are studied in Chapter 4. The construction of some families of balanced incomplete block designs, like Steiner triple systems and Kirkman triple systems, appears in Chapter 6, and as a preparation pairwise balanced designs and group divisible designs are introduced in Chapter 5. t-designs and partially balanced incomplete block designs, as generalizations of balanced incomplete block designs, are studied in Chapters 7, 8 and Chapter 9, respectively.

The author is mostly grateful to Professor Rodney Roberts of Florida State University, Professor Shenglin Zhou of South China University of Technology and Professor Lie Zhu of Soochow University, who read the manuscript carefully, pointed out many typos and give valuable suggestions. Professor Zhou also prepared the bibliography and exercises for the book. Finally, the author is also indebted to the graduate students Jun-Wei Guo, Yan-Ping Mu, Yun Qin, Yi-Dong Sun, Chao Wang, De-Heng Xu, Li-Bo Yang and Li-Ming Yang of the Center for Combinatorics of Nankai University who typed the book except the last Chapter.

Zhe-Xian Wan
2009 Beijing

Contents

Chapter 1

*BIBD*s

1.1 Definition and Fundamental Properties of *BIBD*s

Definition 1.1 *Let v, k and λ be integers such that $v \geq k \geq 2$ and $\lambda \geq 1$. Let X be a finite set of elements, called points, and let $\mathcal{B}$ be a finite collection of subsets of X, called blocks. The pair $(X, \mathcal{B})$ is called a (v, k, λ) balanced incomplete block design or, simply, a (v, k, λ)-BIBD, if the following conditions hold:*

(i) $|X| = v$.

(ii) $|B| = k$ *for all* $B \in \mathcal{B}$.

(iii) *Every pair of distinct points is contained in exactly λ blocks.*

The set $\{v, k, \lambda\}$ is called the set of parameters of the BIBD $(X, \mathcal{B})$. We also use the notation $\mathcal{D} = (X, \mathcal{B})$.

Remark 1.1 A *BIBD* may contain repeated blocks if $\lambda > 1$, which is why we refer to $\mathcal{B}$ as a collection of subsets rather than a set of subsets.

Remark 1.2 If $k = 1$, then we must have $\lambda = 0$; this case is excluded by the assumption $\lambda \geq 1$. Therefore for a (v, k, λ)-*BIBD* we always assume $k \geq 2$ in Definition 1.1. If $v = k$, then every block is equal to X and $(X, \mathcal{B})$ is called a *complete block design*; this is the trivial case. A (v, k, λ)-*BIBD* with $v > k \geq 2$ is said to be *nondegenerate*. In most cases we consider only nondegenerate *BIBD*s.

Example 1.1 *A $(7, 3, 1)$-BIBD.*

$$X = \{1, 2, 3, 4, 5, 6, 7\},$$
$$\mathcal{B} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 7\}\}.$$

In Section 2.3 we will see that $(X, \mathcal{B})$ is a projective plane of order 2.

Example 1.2 *A* $(9,3,1)$-*BIBD*.

$$X = \{1,2,3,4,5,6,7,8,9\},$$
$$\mathcal{B} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\},\{2,5,8\},\{3,6,9\},$$
$$\{1,5,9\},\{2,6,7\},\{3,4,8\},\{1,6,8\},\{2,4,9\},\{3,5,7\}\}.$$

In Section 3.2 we will see that this $(9,3,1)$-*BIBD* is an affine plane of order 3.

Example 1.3 *A* $(16,6,2)$-*BIBD*.

Let $X = \mathbb{Z}_{16} = \{0,1,2,\ldots,15\}$ *and arrange the* 16 *points in the following* 4×4 *array*

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

For each $i \in \mathbb{Z}_{16}$*, let* B_i *be the subset of* $\mathbb{Z}_{16}$ *consisting of the six elements which are situated in the same row or the same column of* i *and are distinct from* i*. For example,* $B_0 = \{1,2,3,4,8,12\}, B_5 = \{1,4,6,7,9,13\}$*, etc. Let* $\mathcal{B} = \{B_i : i \in \mathbb{Z}_{16}\}$*. Clearly* $|B_i| = 6$ *for all* $i \in \mathbb{Z}_{16}$ *and any pair of points is contained in exactly two blocks. For example,* $\{5,10\} \subset B_6, B_9$*. Hence* $(X,\mathcal{B})$ *is a* $(16,6,2)$-*BIBD*.

In Section 2.1 we will see that this $(16,6,2)$-*BIBD* is a symmetric design.

Example 1.4 *Let* X *be a set of* v *points and* $\mathcal{B}$ *consist of all subsets of* X *of size* k*. Then any two points are contained in* $\binom{v-2}{k-2}$ *blocks. Thus* $(X,\mathcal{B})$ *is a* $(v,k,\binom{v-2}{k-2})$-*BIBD*.

Definition 1.2 *Let* $(X,\mathcal{B})$ *be a* (v,k,λ)-*BIBD. Suppose that* $|\mathcal{B}| = b$*. Define a* $v \times b$ 0-1 *matrix*

$$M = (m_{ij})_{1\le i\le v, 1\le j\le b},$$

whose rows are indexed by the points $p_1, p_2, \ldots, p_v$ *and columns are indexed by the blocks* B_1, B_2, ..., B_b, *by*

$$m_{ij} = \begin{cases} 1, & \text{if } p_i \in B_j, \\ 0, & \text{otherwise.} \end{cases}$$

Then M is called the incidence matrix of the BIBD $(X, \mathcal{B})$.

Clearly, the incidence matrix of a *BIBD* depends on the ordering of the points and the ordering of the blocks. For another ordering of points $q_1, q_2, \ldots, q_v$ and another ordering of blocks $C_1, C_2, \ldots, C_b$, the incidence matrix M' of the design takes the form

$$M' = PMQ,$$

where $P = (p_{ij})_{1 \le i,j \le v}$ is a $v \times v$ permutation matrix defined by

$$p_{ij} = \begin{cases} 1, & \text{if } q_i = p_j, \\ 0, & \text{otherwise,} \end{cases}$$

and $Q = (q_{ij})_{1 \le i,j \le b}$ is a $b \times b$ permutation matrix defined by

$$q_{ij} = \begin{cases} 1, & \text{if } B_i = C_j, \\ 0, & \text{otherwise.} \end{cases}$$

Two incidence matrices of the same *BIBD* with respect to different orderings of points and blocks are said to be *equivalent*.

Example 1.5 *In the $(7,3,1)$-BIBD of Example 1.1, let*

$$p_1 = 1,\ p_2 = 2,\ p_3 = 3,\ p_4 = 4,\ p_5 = 5,\ p_6 = 6,\ p_7 = 7$$

and

$$B_1 = \{1,2,3\},\ B_2 = \{1,4,5\},\ B_3 = \{1,6,7\},\ B_4 = \{2,4,7\},$$
$$B_5 = \{2,5,6\},\ B_6 = \{3,4,6\},\ B_7 = \{3,5,7\}.$$

Then the $(7,3,1)$-BIBD has incidence matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

If we let

$$q_1 = 1,\ q_2 = 2,\ q_3 = 3,\ q_4 = 4,\ q_5 = 5,\ q_6 = 7,\ q_7 = 6,$$

then with respect to the ordering of points $q_1, q_2, q_3, q_4, q_5, q_6, q_7$ *and the ordering of blocks as before the* $(7,3,1)$*-BIBD has incidence matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \tag{1.1}$$

which is a symmetric matrix.

Example 1.6 *The incidence matrix of the* $(9,3,1)$*-BIBD of Example 1.2 is the* 9×12 *0-1 matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now we give some basic properties of a (v,k,λ)-*BIBD*.

Theorem 1.1 *In a* (v,k,λ)*-BIBD, every point occurs in exactly*

$$r = \frac{\lambda(v-1)}{k-1} \tag{1.2}$$

blocks.

Proof. By rearranging the points and the blocks, we can assume any given point to be the first point which appears in the first r blocks. Then the incidence matrix takes the form

$$M = \begin{pmatrix} 1\ldots1 & 0\ldots0 \\ M_1 & M_2 \end{pmatrix},$$

where M_1 is a $(v-1)\times r$ matrix and M_2 is a $(v-1)\times(b-r)$ matrix. Count the number of 1's in M_1 in two different ways. On the one hand, M_1 has r columns and each column has $k-1$ 1's. On the other hand, M_1 has $v-1$ rows and each row has λ 1's. Therefore, $r(k-1) = \lambda(v-1)$, which implies (1.2). ∎

By Theorem 1.1 the number of blocks containing any given point in a (v,k,λ)-$BIBD$ is a constant, which is denoted by r and is called the *replication number* of the $BIBD$.

Theorem 1.2 *A (v,k,λ)-$BIBD$ has exactly*

$$b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k} \tag{1.3}$$

blocks.

Proof. Let M be the incidence matrix of the (v,k,λ)-$BIBD$. Count the number of 1's in M in two different ways. Firstly, M has b columns and each column has k 1's. Secondly, M has v rows and by Theorem 1.1, each row has r 1's. Thus $bk = vr$, which implies $b = \frac{vr}{k}$. Substituting (1.2) into it, we obtain $b = \frac{\lambda(v^2-v)}{k^2-k}$. ∎

A (v,k,λ)-$BIBD$ is also called a (v,b,r,k,λ)-$BIBD$, where $r = \frac{\lambda(v-1)}{k-1}$ and $b = \frac{vr}{k}$, and $\{v,b,r,k,\lambda\}$ is also called the set of its parameters.

Theorem 1.3 *Let M be a $v \times b$ 0-1 matrix. Then M is the incidence matrix of a (v,k,λ)-$BIBD$ if and only if both*

$$M\,{}^tM = \lambda J_v + (r-\lambda)I_v \tag{1.4}$$

and

$$1_v M = k 1_b \tag{1.5}$$

hold, where ${}^tM$ denotes the transpose of M, $r = \lambda(v-1)/(k-1)$, J_v and I_v are the $v \times v$ all 1's matrix and the identity matrix, respectively, and 1_v and 1_b are the v-dimensional and b-dimensional all 1 row vectors, respectively.

Proof. First, let M be the incidence matrix of a (v,k,λ)-$BIBD$ $(X,\mathcal{B})$ and let $X = \{p_1, \ldots, p_v\}$ and $\mathcal{B} = \{B_1, \ldots, B_b\}$. Then the (i,j)-entry of $M\,{}^tM$ is

$$\sum_{k=1}^{b} m_{ik}m_{jk} = \begin{cases} r, & \text{if } i = j, \\ \lambda, & \text{if } i \neq j. \end{cases}$$

Hence every entry on the main diagonal of $M\,{}^tM$ is equal to r and every off-diagonal entry is equal to λ, so $M\,{}^tM = \lambda J_v + (r-\lambda)I_v$.

Moreover, the i-th entry of $1_v M$ is equal to the number of 1's in the i-th column of M, which is equal to the size of the i-th block, and hence, is equal to k. Therefore $1_v M = k1_b$.

Conversely, suppose M is a $v \times b$ 0-1 matrix which satisfies (1.4) and (1.5). Let $X = \{p_1, \ldots, p_v\}$ and

$$M = (m_{ij})_{1 \le i \le v, 1 \le j \le b}.$$

Define

$$B_j = \{p_i \in X : m_{ij} = 1\}, \quad j = 1, 2, \ldots, b$$

and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. By (1.5), there are k 1's in every column of M, so $|B_i| = k$ for all $i = 1, 2, \ldots, b$. From (1.4) it follows that every pair of distinct points is contained in exactly λ blocks. Therefore $(X, \mathcal{B})$ is a (v, k, λ)-$BIBD$ with M as its incidence matrix. ∎

Example 1.7 *Let A be the incidence matrix of the $(9, 3, 1)$-$BIBD$ of Example 1.2, viz., A is the matrix given in Example 1.6. Let*

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$D = J - C,$$

where J is the 3×12 all 1 matrix. Then it can be verified directly that

$$M = \begin{pmatrix} {}^tA & {}^tA & {}^tC & {}^tD \\ B & J - B & 0 & 0 \end{pmatrix}$$

is the incidence matrix of a $(16, 6, 3)$-$BIBD$.

Theorem 1.4 (Fisher's Inequality) *In any nondegenerate (v, b, r, k, λ)-$BIBD$, $b \ge v$.*

Proof. Let M be the incidence matrix of the $BIBD$. By Theorem 1.3

$$M\,{}^tM = \lambda J_v + (r - \lambda) I_v.$$

Let us calculate $\det(M\,{}^tM)$.

$$
\begin{aligned}
&\det(M^{\,t}M)\\
&= \det\begin{pmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \lambda & \lambda & r & \dots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \dots & r \end{pmatrix}\\
&= \det\begin{pmatrix} r+\lambda(v-1) & r+\lambda(v-1) & r+\lambda(v-1) & \dots & r+\lambda(v-1) \\ \lambda & r & \lambda & \dots & \lambda \\ \lambda & \lambda & r & \dots & \lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda & \lambda & \lambda & \dots & r \end{pmatrix}\\
&= (r+\lambda(v-1))\det\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda & r & \lambda & \dots & \lambda \\ \lambda & \lambda & r & \dots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \dots & r \end{pmatrix}\\
&= (r+\lambda(v-1))\det\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda & r-\lambda & 0 & \dots & 0 \\ \lambda & 0 & r-\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & 0 & 0 & \dots & r-\lambda \end{pmatrix}\\
&= (r+\lambda(v-1))(r-\lambda)^{v-1}.
\end{aligned}
$$

Since $v \neq 1$, we have $r + \lambda(v-1) \neq 0$. Since the *BIBD* is nondegenerate, we have $v > k$ and by Theorem 1.1 $r > \lambda$. Therefore $\det(M^{\,t}M) \neq 0$, which implies $b \geq v$. ∎

Theorems 1.1, 1.2 and 1.4 are necessary conditions for the existence of a (v, k, λ)-*BIBD*. We can use them to exclude some parameter sets to be the parameter sets of *BIBD*s as the following examples show.

Example 1.8 *There does not exist an* $(8, 3, 1)$-*BIBD, for*

$$r = \frac{\lambda(v-1)}{k-1} = \frac{7}{2} \notin \mathbb{Z}.$$

Example 1.9 *There does not exist a* $(19, 4, 1)$-*BIBD, for*

$$r = \frac{\lambda(v-1)}{k-1} = 6, \text{ but } b = \frac{vr}{k} = \frac{19\cdot 3}{2} \notin \mathbb{Z}.$$

Example 1.10 *There does not exist a* $(16, 6, 1)$-*BIBD, for*

$$r = \frac{\lambda(v-1)}{k-1} = 3, \text{ but } b = \frac{vr}{k} = 8 < v = 16.$$

One of the main goals of combinatorial design theory is to determine necessary and sufficient conditions of the parameter set $\{v, k, \lambda\}$ for the existence of a (v, k, λ)-*BIBD*. This is a very difficult problem in general, and there are many parameter sets in which the answers are not yet known. For example, it is currently unknown if there exists a (22,8,4)-*BIBD* (such a *BIBD* would have $r = 12$ and $b = 33$). On the other hand, there are many known constructions for infinite classes of *BIBD*s, as well as some other necessary conditions, which will be discussed a bit later.

1.2 Isomorphisms and Automorphisms

Definition 1.3 *Let* $(X, \mathcal{B})$ *and* $(Y, \mathcal{C})$ *be two BIBDs. If there is a bijective map* $\alpha : X \to Y$ *and a bijective map* $\alpha : \mathcal{B} \to \mathcal{C}$ *such that for all* $x \in X$ *and* $B \in \mathcal{B}$, $x \in B$ *if and only if* $\alpha(x) \in \alpha(\mathcal{B})$, *then* $(X, \mathcal{B})$ *and* $(Y, \mathcal{C})$ *are said to be isomorphic and* α *is called an isomorphic map or an isomorphism. (Note that we use the same symbol* α *to denote both the map* $X \to Y$ *and the map* $\mathcal{B} \to \mathcal{C}$.)

If $(X, \mathcal{B})$ *and* $(Y, \mathcal{C})$ *are isomorphic BIBDs, we write* $(X, \mathcal{B}) \simeq (Y, \mathcal{C})$.

Example 1.11 *Consider the* $(4, 2, 2)$-*BIBDs* $(X, \mathcal{B})$ *and* $(Y, \mathcal{C})$.

$$X = \{1, 2, 3, 4\},$$
$$\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11}, B_{12}\},$$

where

$$B_1 = B_2 = \{1, 2\}, \ B_3 = B_4 = \{3, 4\}, \ B_5 = B_6 = \{1, 3\},$$
$$B_7 = B_8 = \{2, 4\}, \ B_9 = B_{10} = \{1, 4\}, \ B_{11} = B_{12} = \{2, 3\},$$

and

$$Y = \{a, b, c, d\},$$
$$\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}\},$$

where

$$C_1 = C_2 = \{a, b\}, \ C_3 = C_4 = \{c, d\}, \ C_5 = C_6 = \{a, c\},$$
$$C_7 = C_8 = \{b, d\}, \ C_9 = C_{10} = \{a, d\}, \ C_{11} = C_{12} = \{b, c\}.$$

Define a bijection $\alpha : X \to Y$ *by* $\alpha(1) = a$, $\alpha(2) = b$, $\alpha(3) = c$, $\alpha(4) = d$ *and a bijection* $\alpha : \mathcal{B} \to \mathcal{C}$ *by* $\alpha(B_1) = C_2$, $\alpha(B_2) = C_1$, *and* $\alpha(B_i) = C_i, 3 \leqslant i \leqslant 12$. *Then* α *is an isomorphism from* $(X, \mathcal{B})$ *to* $(Y, \mathcal{C})$.

If $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are two $BIBD$s without repeated blocks, we also adopt the following definition of isomorphism.

Definition 1.4 *Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be two $BIBD$s without repeated blocks. If there exists a bijection $\alpha : X \to Y$ such that*

$$\{\{\alpha(x) : x \in B\} : B \in \mathcal{B}\} = \mathcal{C},$$

then $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are said to be isomorphic and α is called an isomorphic map or an isomorphism. In other words, if we rename every point $x \in X$ by $\alpha(x)$, then the collection of blocks $\mathcal{B}$ is transformed into $\mathcal{C}$.

Clearly, when $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are two $BIBD$s without repeated blocks, Definitions 1.3 and 1.4 are equivalent.

Example 1.12 *Here are two $(7,3,1)$-$BIBD$s $(X, \mathcal{B})$ and $(Y, \mathcal{C})$:*

$X = \{1, 2, 3, 4, 5, 6, 7\}$,
$\mathcal{B} = \{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,7\}, \{2,5,6\}, \{3,4,6\}, \{3,5,7\}\}$

and

$Y = \{a, b, c, d, e, f, g\}$,
$\mathcal{C} = \{\{a,b,d\}, \{a,c,e\}, \{a,f,g\}, \{b,c,g\}, \{b,e,f\}, \{c,d,f\}, \{d,e,g\}\}$.

Define a bijection $\alpha : X \to Y$ by $\alpha(1) = a, \alpha(2) = b, \alpha(3) = d, \alpha(4) = c, \alpha(5) = e, \alpha(6) = f, \alpha(7) = g$. Clearly, $\mathcal{B}$ is transformed to $\mathcal{C}$ by α. Hence $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are isomorphic and α is an isomorphism.

If we define $\beta : X \to Y$ by $\beta(1) = a, \beta(2) = g, \beta(3) = f, \beta(4) = c, \beta(5) = e, \beta(6) = d, \beta(7) = b$. It can also be verified that $\mathcal{B}$ is transformed to $\mathcal{C}$ by β. Thus β is another isomorphism.

Clearly, isomorphic $BIBD$s have the same parameter set, and we usually do not distinguish isomorphic $BIBD$s. It is left as an exercise (Exercise 1.5) to show that there is only one (7,3,1)-$BIBD$ up to an isomorphism. In general, it is a difficult computation problem to determine whether two $BIBD$s with the same parameter set are isomorphic . There are $v!$ possible bijections between two sets of cardinality v. To identify that two (v, k, λ)-$BIBD$s are not isomorphic, we must show that none of the possible $v!$ bijections is an isomorphism. Since $v!$ grows exponentially quickly as a function of v, it soon becomes impractical to actually test every possible bijection. Thus we have to try to find more sophisticated algorithms rather than testing every possibility exhaustively.

Isomorphism of $BIBD$s can also be described in terms of incidence matrices.