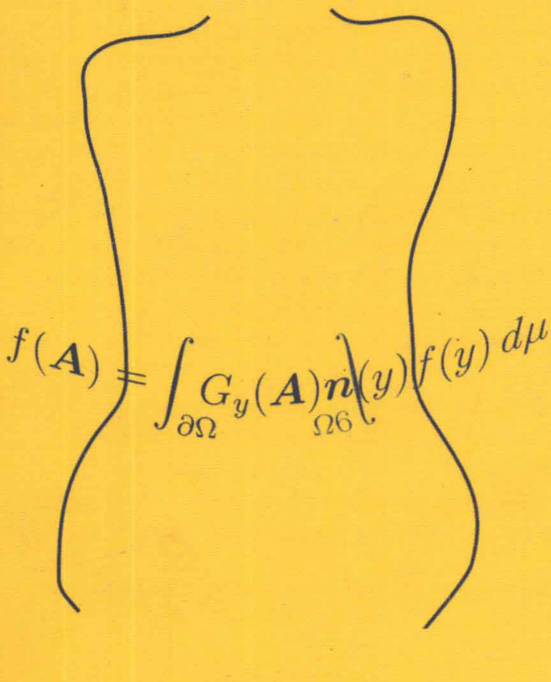
Lecture Notes in Mathematics

Brian Jefferies

Spectral Properties of Noncommuting Operators

1843



Springer

Brian Jefferies

Spectral Properties of Noncommuting Operators

Springer

Author

Brian Jefferies

School of Mathematics
University of New South Wales
Sydney, NSW, 2052
Australia

e-mail: b.jefferies@unsw.edu.au
http://www.maths.unsw.edu.au/~brianj

Library of Congress Control Number: 2004104471

Mathematics Subject Classification (2000):
47A13, 47A60, 30G35, 42B20

ISSN 0075-8434
ISBN 3-540-21923-4 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag is a part of Springer Science + Business Media

http://www.springeronline.com

Typesetting: Camera-ready TEX output by the authors

SPIN: 11001249 41/3142/du - 543210 - Printed on acid-free paper

Lecture Notes in Mathematics

Edited by J.-M. Morel, F. Takens and B. Teissier

Editorial Policy
for the publication of monographs

1. Lecture Notes aim to report new developments in all areas of mathematics and their applications- quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome.

   Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They may be based on specialised lecture courses. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for doctoral theses to be accepted for the Lecture Notes series, though habilitation theses may be appropriate.

2. Manuscripts should be submitted (preferably in duplicate) either to Springer's mathematics editorial in Heidelberg, or to one of the series editors (with a copy to Springer). In general, manuscripts will be sent out to 2 external referees for evaluation. If a decision cannot yet be reached on the basis of the first 2 reports, further referees may be contacted: The author will be informed of this. A final decision to publish can be made only on the basis of the complete manuscript, however a refereeing process leading to a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter, a bibliography and several sample chapters.

   Authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer refereeing times and nevertheless unclear referees' recommendations, making further refereeing of a final draft necessary.

   Authors should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

3. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should always include
   - a table of contents;
   - an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
   - a subject index: as a rule this is genuinely helpful for the reader.

Continued on inside back-cover

Lecture Notes in Mathematics 1843

Springer
Berlin
Heidelberg
New York
Hong Kong
London
Milan
Paris
Tokyo

Preface

The work described in these notes has had a long gestation. It grew out of my sojourn at Macquarie University, Sydney, 1986-87 and 1989-90, during which time Alan McIntosh was applying Clifford analysis techniques to the study of singular integral operators and irregular boundary value problems. His research group provided a stimulating and convivial environment over the years. I would like to thank my collaborators in this enterprise: Jerry Johnson, Alan McIntosh, Susumu Okada, James Picton-Warlow, Werner Ricker, Frank Sommen and Bernd Straub. The work was supported by two large grants from the Australian Research Council.

Sydney, March 2004 *Brian Jefferies*

Contents

1

Introduction

The subject of these notes is the spectral theory of systems of operators. Because 'spectral theory' means different things to different workers in functional analysis, it is worthwhile to first set down how the term is used in the present context and the relationship it bears to the spectral theory of a single selfadjoint operator.

The *spectrum* $\sigma(A)$ of a single matrix A is the finite set of all *eigenvalues* of A, that is, complex numbers λ for which the equation $Av = \lambda v$ has a nonzero vector v as a solution. In order to treat linear operators A acting on some function space, it is preferable to take $\sigma(A)$ to mean the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not invertible. The most complete spectral analysis is available for selfadjoint operators A acting in Hilbert space, for then the linear operator A has a spectral decomposition

$$A = \int_{\sigma(A)} \lambda \, dP_A(\lambda) \tag{1.1}$$

with respect to a spectral measure P_A associated with A. In the case that A is an hermitian matrix, the integral representation (1.1) becomes a finite sum

$$A = \sum_{\lambda \in \sigma(A)} \lambda \, P_A(\{\lambda\}) \tag{1.2}$$

in which $P_A(\{\lambda\})$ is the orthogonal projection onto the eigenspace of the eigenvalue λ. The spectral theory of selfadjoint operators lies at the foundation of quantum physics.

The solution of linear operator equations, such as those that arise in quantum mechanics, often requires the formation of functions of operators. For example, in order to solve the linear equation

$$\frac{du(t)}{dt} + Au(t) = 0, \quad u(0) = u_0,$$

we need to form the exponential e^{-tA}, $t \geq 0$, of A. Because of the importance of linear evolution equations, the theory of exponentiating an operator is well-understood, but in general, the *spectral* properties of A determine the types of functions $f(A)$ of A that can be formed in a reasonable manner.

In the case of a selfadjoint operator A, we can take

$$f(A) = \int_{\sigma(A)} f(\lambda)\, dP_A(\lambda) \tag{1.3}$$

for any P_A-essentially bounded Borel measurable function $f : \sigma(A) \to \mathbb{C}$. The pleasant spectral properties of a selfadjoint operator A are reflected in the rich class of functions $f(A)$ of A that can be formed.

A basic task of quantum mechanics is to find a quantum representation $f(P,Q)$ of a classical observable $(p,q) \longmapsto f(p,q)$ on phase space. Here $P = \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}$ is the momentum operator and Q is the position operator of 'multiplication by x'. They satisfy the commutation relation $QP - PQ = i\hbar I$. For example, if $H(p,q) = \frac{p^2}{2m} + V(q)$ is the classical hamiltonian of the system, then $H(P,Q) = \frac{P^2}{2m} + V(Q)$ is the corresponding quantum observable, provided that the sum of the two unbounded operators is interpreted appropriately. Although it is known that the structure of classical observables is not preserved in the quantum setting for an extensive class of observables f, we are left with the problem of forming a function $f(P,Q)$ of a pair (P,Q) of operators which do not commute with each other.

In another context, symmetric hyperbolic systems

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} = 0 \tag{1.4}$$

of partial differential equations arise in the linearised equations of magnetohydrodynamics [15]. In the case that the matrices $A_1, \dots, A_n$ are hermitian, the fundamental solution is the matrix-valued distribution

$$\frac{1}{(2\pi)^n}\left(e^{it\sum_{j=1}^{n} A_j\xi_j}\right)^{\hat{}}.$$

Here the Fourier transform $\hat{}$ is taken in the sense of distributions with respect to the variable $\xi \in \mathbb{R}^n$.

Then the fundamental solution $f \longmapsto f(A_1, \dots, A_n)$ of (1.4) at time $t = 1$ may be viewed as a mapping that forms functions $f(A_1, \dots, A_n)$ of the n matrices $A_1, \dots, A_n$. The snapshot of the support of the fundamental solution at time $t = 1$ determines the propagation cone of solutions of the initial value problem for the symmetric hyperbolic system (1.4). A mapping such as $f \longmapsto f(A_1, \dots, A_n)$ will be termed a *functional calculus* in this work. Although the expression is used somewhat loosely, the idea is common to the areas in functional analysis just mentioned.

In the traditional setting of a single operator A, a decent functional calculus $f \longmapsto f(A)$ is a *homomorphism* of Banach algebras: $(fg)(A) = f(A)g(A)$

for two functions f, g belonging to the domain of the functional calculus. In the case of a selfadjoint operator A, the domain of the functional calculus defined by formula (1.3) is the Banach algebra $L^\infty(P_A)$ under pointwise multiplication. For two operators A_1, A_2 which do not commute, there is a choice in operator ordering. For example, given the function $f(z_1, z_2) = z_1 z_2$, the operator $f(A_1, A_2)$ could be A_1A_2, A_2A_1, $\frac{1}{2}(A_1A_2 + A_2A_1)$ or some other choice of weighted operator product. Under these circumstances, the homomorphism property fails, but we still use the term 'functional calculus'.

In the noncommutative setting of spectral theory considered in the present work, there is a shift of emphasis from the algebraic formulation of the spectrum to a more analytic formulation of the 'joint spectrum' of operators $(A_1, \dots, A_n)$ as the underlying set on which the 'richest' functional calculus $f \longmapsto f(A_1, \dots, A_n)$ is defined. From this point of view, the 'joint spectrum' of matrices $(A_1, \dots, A_n)$ associated with the symmetric hyperbolic system (1.4) determines the propagation cone of the solution, so it has a natural interpretation. For bounded selfadjoint operators, the 'joint spectrum' of $(A_1, \dots, A_n)$ can be defined algebraically in terms of commutative objects $(\tilde{A}_1, \dots, \tilde{A}_n)$ associated with $(A_1, \dots, A_n)$, see Section 7.1.

The study of functions of noncommuting operators has been extensively developed by V.P. Maslov and co-workers, see [82] for a list of references. The calculus of noncommuting operators has fundamental applications to the asymptotic analysis of differential equations, quantisation and quantum groups. The emphasis in the present work is in a different direction: the properties of the support of functional calculi associated with the operators $(A_1, \dots, A_n)$ is examined and the relationship between the nature of the operators $(A_1, \dots, A_n)$ and possible functional calculi is explored. In the case of a single operator, this is the traditional domain of *spectral theory*. The support of the 'natural' functional calculus is interpreted as the joint spectrum of the operators $(A_1, \dots, A_n)$ and it is in this sense that the work is devoted to the spectral properties of systems of noncommuting operators.

Even for a single bounded selfadjoint operator A, there is a choice between the 'richest' functional calculus $f \longmapsto f(A)$ for $f \in L^\infty(P_A)$ and the functional calculus $f(A) = \sum_{j=0}^\infty c_j A^j$ for functions f with a uniformly convergent power series expansion $\sum_{j=0}^\infty c_j z^j$ for all $z \in \mathbb{C}$ belonging to the closed unit disk $D(r)$ of radius $r = \|A\|$ centred at zero. The spectrum $\sigma(A)$ of A is precisely the support of the richest functional calculus rather than the closed disk $D(r)$ – a set much larger than $\sigma(A)$.

When the operators $(A_1, \dots, A_n)$ commute with each other, a general notion of joint spectrum relies on ideas from algebraic topology [104], [111], [25]. However, for the class of operators treated in this work, such considerations are unnecessary (see [76] for a comparison of joint spectra in the commuting case) and we can deal with both the commutative and noncommutative setting simultaneously. Of course, this is at the expense of placing a restriction on the combined spectra of the operators $(A_1, \dots, A_n)$, which should be on (or, in Chapter 6, not be too far from) the real axis. Recent work [10] shows how this restriction can be lifted in the Hilbert space setting. Certain ideas from algebraic geometry do play a part in Chapter 5 in the context of computing the joint spectrum of a system of hermitian matrices.

The subject of these notes is the generalisation of the spectral theory of a single operator to the setting of a finite system of possibly noncommuting bounded (or, in Chapter 6, densely defined) operators. There are other means by which the program can be realised. The noncommuting variable approach is taken in a series of papers by J.L. Taylor [104, 105, 106, 107]. A geometric approach in von Neumann algebras is taken in [2]. One could attempt to compute the Gelfand spectrum of corresponding commuting objects, see [83], [3], [4], [6] and Section 7.1 below. A monograph surveying many results in several variable spectral theory has recently appeared [80].

Another point of view is to see to what extent the Spectral Mapping Theorem for a single operator generalises to a system of operators, especially with weak commutativity assumptions – see [74] and [36, 37] for this approach.

It should be obvious from the description above that the present mathematical work has its roots in physical applications. Indeed, the spectral theory of a single selfadjoint operator was developed by J. von Neumann [113] in order to put quantum mechanics on a firm foundation. The names of the mathematician H. Weyl and the physicist R. Feynman recur in this work. Both were motivated by problems in quantum physics.

In [115], H. Weyl proposed the functional calculus

$$\frac{1}{2\pi}\left(e^{i\xi_1 P+i\xi_2 Q}\right)^{\hat{}} : f \longmapsto f(P,Q)$$

as a quantisation procedure sending the classical observable f on phase space to the quantum observable $f(P,Q)$. Although a real valued function is mapped to a selfadjoint operator, a nonnegative observable need not be mapped to a positive operator, that is, a quantum observable whose expectation values are nonnegative; from this point of view, the procedure is physically unrealistic except for a limited class of classical observables.

An operational calculus for systems of noncommuting operators was proposed by R. Feynman [28] with a view of applications to quantum electrodynamics. The idea is to attach time indices to the operators concerned, treat the resulting operator valued functions as commuting objects in functional calculations and, at the end of the day, 'disentangle' the resulting expressions by restoring time-ordering in which operators with earlier time indices than other operators act *first*. The connection with Weyl's calculus was fleshed out by E. Nelson [83].

A natural approach to forming functions $f(A)$ of a single bounded linear operator A is to apply the Riesz-Dunford formula

$$f(A) = \frac{1}{2\pi i}\int_C (\zeta I - A)^{-1} f(\zeta)\, d\zeta \tag{1.5}$$

to a function f holomorphic in a neighbourhood of the spectrum $\sigma(A)$ of A and a suitable closed contour C about $\sigma(A)$. Although this approach can be generalised to systems $A = (A_1, \ldots, A_n)$ of commuting bounded linear operators and holomorphic functions defined in $\mathbb{C}^n$ by defining the joint spectrum in terms of the Koszul complex [104], [111], a different line is taken in these notes.

Clifford analysis also possesses an analogue of the Cauchy integral formula in one complex variable for higher dimensions. The Clifford algebra $\mathbb{C}_{(n)}$ is a complex algebra with unit e_0, generated by n anti-commuting vectors $e_1, \ldots, e_n$. A function $f(x_0, x_1, \ldots, x_n)$ of $n+1$ real variables $x_0, x_1, \ldots, x_n$, with values in $\mathbb{C}_{(n)}$ and satisfying $Df = 0$ for the operator

$$D = \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}$$

is called *left monogenic*. The Cauchy integral formula takes the form

$$f(x) = \int_{\partial\Omega} G_y(x) \boldsymbol{n}(y) f(y)\, d\mu(y), \quad x \in \Omega. \tag{1.6}$$

Here f is left monogenic in a neighbourhhood of $\overline{\Omega}$, where Ω is a bounded open subset of $\mathbb{R}^{n+1}$ with smooth oriented boundary $\partial\Omega$ and outward unit normal $\boldsymbol{n}(y)$ at $y \in \partial\Omega$. The surface measure of $\partial\Omega$ is denoted by μ. The Cauchy kernel

$$G_y(x) = \frac{1}{\Sigma_n} \frac{\overline{y - x}}{|y - x|^{n+1}}, \quad x, y \in \mathbb{R}^{n+1}, x \neq y, \tag{1.7}$$

with $\Sigma_n = 2\pi^{\frac{n+1}{2}} / \Gamma\left(\frac{n+1}{2}\right)$ the volume of unit n-sphere in $\mathbb{R}^{n+1}$, is the analogue of the normalised Cauchy kernel $\frac{1}{2\pi}(\zeta - z)^{-1}$ in complex analysis. The theme of the present notes is to form functions $f(A_1, \ldots, A_n)$ of n operators $A_1, \ldots, A_n$ via the formula

$$f(A_1, \ldots, A_n) = \int_{\partial\Omega} G_y(A_1, \ldots, A_n) \boldsymbol{n}(y) f(y)\, d\mu(y), \tag{1.8}$$

which arises by analogy with the Riesz-Dunford formula (1.5). The principal difficulty is making sense of the function $x \longmapsto G_x(A_1, \ldots, A_n)$ and determining its singularities, the collection of which may be viewed as the *joint spectrum* of the system $(A_1, \ldots, A_n)$ of operators. Along the way to realising this idea, we shall make contact with the Weyl functional calculus for n operators, Feynman's operational calculus and the fundamental solution of the symmetric hyperbolic system (1.4).

It may seem somewhat surprising that Clifford analysis should be a tool in the analysis of the spectral theory of systems of operators. These notes grew out of a desire to bring together the seemingly disparate streams of thought I have been exposed to over the years by my friends and colleagues. On the one hand, A. McIntosh has been an enthusiastic proponent of Clifford techniques in harmonic analysis and the solution of irregular boundary value problems in partial differential equations [73]. A connection with Weyl's calculus appears in joint work with A. Pryde [75], [87], [88], [89]. On the other hand, the joint work of G.W. Johnson and M. Lapidus [59], [60], [61] and G.W. Johnson with myself [48], [49], [50], [51] shows the connection of Feynman's operational calculus with the monogenic functional calculus for systems of operators described in these notes.

Feynman viewed his operational calculus as a procedure to invoke when the *Feynman integral*, as such, cannot be applied. Indeed, there is an allusion to Clifford analysis techniques in [28, Appendix B, p. 126]: *The Pauli matrices (times i) are the basis for the algebra of quaternions so that the solution of such problems* [concerning functional calculi] *might open up the possibility of a true infinitesimal calculus of quantities in the field of hypercomplex numbers.* In the point of view set out here, for the Pauli matrices σ_1, σ_2, σ_3, the key property needed for the construction of a joint functional calculus by the method of these notes is that they are *selfadjoint*, so that $\xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3$ has real spectrum for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Even if the n operators $A_1, \dots, A_n$ do not have real spectra, it is enough to require that the spectrum of the operator $\sum_{j=1}^n \xi_j A_j$ is contained in a fixed sector in $\mathbb{C}$ for all $\xi \in \mathbb{R}^n$ in order that the functional calculus described here should exist.

It is by utilising the underlying real-variable characteristics of Clifford analysis of monogenic functions defined in $\mathbb{R}^{n+1}$ and the spectral properties of the operators $A_1, \dots, A_n$ that we can bypass homological considerations of [104], [111], [25], leading to a rather straightforward approach to forming functions of systems of noncommuting operators. Even in this restricted setting, there is considerable scope for investigating the properties of joint functional calculi and their relationship with quantisation procedures and the geometric analysis of the support of solutions of the hyperbolic system (1.4) of partial differential equations.

A more detailed description of the contents of the present notes and the connection with the work of these authors follows.

The background to Weyl's functional calculus is given in Section 1 of Chapter 2. A unitary representation of the Heisenberg group is used to form functions $\sigma(\boldsymbol{D}, \boldsymbol{X})$ of position $\boldsymbol{X}$ and momentum operators $\boldsymbol{D}$ in quantum mechanics on $\mathbb{R}^n$. The same idea works for a system $\boldsymbol{A} = (A_1, \dots, A_n)$ of n bounded linear operators on a Banach space provided that the right exponential growth estimates (2.2) are satisfied and this is described carefully in Section 2 of Chapter 2, from work of E. Nelson [83], M. Taylor [108], R.F.V. Anderson [7], [8] and A. Pryde [88]. The *joint spectrum* $\gamma(\boldsymbol{A})$ of $\boldsymbol{A}$ is simply the support of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$ – an operator valued distribution with compact support.

For $n = 1$, a single bounded linear operator A satisfies the exponential growth estimate (2.2) precisely when it is a *generalised scalar operator* with real spectrum [23]. Such operators may be viewed as generalisations of self-adjoint operators for which spectral measures are replaced by spectral distributions.

Chapter 3 sets down the background in Clifford analysis, such as the Cauchy integral formula (1.6), needed to construct a functional calculus for operators. Most of the material here is from the monograph [19]. Other important formulae include the monogenic representation of distributions (Theorem 3.3) and the plane wave decomposition of the Cauchy kernel (Proposition 3.4). Proposition 3.6 gives an approximation result for real analytic functions with a proof due to F. Sommen.

A natural way to construct the Cauchy kernel for an n-tuple $\boldsymbol{A} = (A_1, \dots, A_n)$ of mutually commuting operators with real spectra is to adapt formula (1.7) in the time-honoured way by replacing the vector $x \in \mathbb{R}^n$ by the n-tuple $\boldsymbol{A}$ and writing

$$G_y(\boldsymbol{A}) = \frac{1}{\Sigma_n}\left(\overline{y} + \sum_{j=1}^{n} A_j e_j\right)\left(y_0^2 I + \sum_{j=1}^{n}(y_j I - A_j)^2\right)^{-(n+1)/2}, \tag{1.9}$$

for all $y = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$ with $y_0 \neq 0$. Then the Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$ will have singularities on the set

$$\gamma(\boldsymbol{A}) = \left\{(0, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : 0 \in \sigma\left(\sum_{j=1}^{n}(y_j I - A_j)^2\right)\right\}. \tag{1.10}$$

This is the basic idea of the paper [75] of A. McIntosh and A. Pryde. If n is odd, then formula (1.9) is readily interpreted and a functional calculus may be constructed via the Riesz-Dunford formula (1.8). If n is even, it is not clear how the fractional power should be interpreted.

Chapter 4 examines this problem from two viewpoints. If $\boldsymbol{A}$ satisfies the exponential growth estimates (2.2), then $G_y(\boldsymbol{A})$ may be defined as $\mathcal{W}_{\boldsymbol{A}}(G_y)$ for all $y \in \mathbb{R}^{n+1}$ outside the support $\gamma(\boldsymbol{A})$ of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$. The observation that the operator valued distribution $\mathcal{W}_{\boldsymbol{A}}$ may be passed from outside the Clifford version of the Cauchy integral formula (1.6) into the integrand verifies the Riesz-Dunford formula (1.8). It is proved in Theorem 4.8 that $\gamma(\boldsymbol{A})$ is exactly the set of singularities of the Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$. Section 4.1 is based on [53]. Unlike formula (1.9), it is not necessary to assume that $\boldsymbol{A}$ consists of commuting operators.

On the other hand, the original motivation for the study of the representation (1.8) was to treat the (commuting) unbounded operators of differentiation on a Lipschitz surface – a system of operators that does not satisfy the exponential estimates (2.2). Soon after the work [75], A. McIntosh realised that the plane wave decomposition of the Cauchy kernel [103] could be used profitably in the present context. Sections 4.2 and 4.3 are based on joint work [54] of the author with A. McIntosh and J. Picton-Warlow and represent the

Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$ in terms of the plane wave formula. Rather than the the exponential estimates (2.2), what is essential here is the condition (4.10) that real linear combinations of $A_1, \ldots, A_n$ should have real spectra. It is not necessary to assume that the bounded linear operators $A_1, \ldots, A_n$ commute with each other. Now the joint spectrum $\gamma(\boldsymbol{A})$ is *defined* to be the set of singularities of the Cauchy kernel $y \longmapsto G_y(\boldsymbol{A})$.

A basic property of the notion of a 'spectrum' of an operator or system of operators is that disjoint components should be associated with projections onto subspaces left invariant by the system. That the joint spectrum $\gamma(\boldsymbol{A})$ enjoys this property is proved in Section 4.4 by appealing to formula (1.8). The result is actually a consequence of a general version of the noncommutative Shilov idempotent theorem [4, Theorem 4.1] proved by E. Albrecht, but the Clifford analysis techniques used in Section 4.4 are natural in the present context.

Chapter 5 exploits the complementary viewpoints of the joint spectrum $\gamma(\boldsymbol{A})$ for a system $\boldsymbol{A} = (A_1, \ldots, A_n)$ of *matrices* as the set of singularities of the Cauchy kernel $G_{(\cdot)}(\boldsymbol{A})$ and as the support of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$. For matrices, the spectral reality condition (4.10) is equivalent to the exponential growth estimates (2.2) necessary for the existence of the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$. This is proved in Section 5.2 following [44] although, in another language, the result is known from the techniques of partial differential equations, see [58, p. 153]. An explicit formula for $\mathcal{W}_{\boldsymbol{A}}$ due to E. Nelson [83, Theorem 9] is proved in Section 5.1 for the case that $A_1, \ldots, A_n$ are hermitian $N \times N$ matrices. The proof is based on [42].

The 'numerical range' of the system $\boldsymbol{A}$ enters into Nelson's formula. Let $S(\mathbb{C}^N) = \{u \in \mathbb{C}^N : |u| = 1\}$ be the unit sphere in $\mathbb{C}^N$. The numerical range map $W_{\boldsymbol{A}} : S(\mathbb{C}^N) \to \mathbb{R}^n$ is defined by

$$W_{\boldsymbol{A}} : u \longmapsto (\langle A_1 u, u \rangle, \ldots, \langle A_n u, u \rangle), \quad u \in S(\mathbb{C}^N),$$

with $\langle \cdot, \cdot \rangle$ representing the inner product of $\mathbb{C}^N$. The range of $W_{\boldsymbol{A}}$ is the 'generalised numerical range' of the system $\boldsymbol{A}$. For the case $n = 2$, the range of the map $W_{\boldsymbol{A}}$ is just the usual numerical range of the $(N \times N)$ matrix $A_1 + iA_2$. Differential properties of the numerical range map and their relationship to spectral properties of the matrix $A_1 + iA_2$ are studied in [38] and [63]. The matrix valued distribution $\mathcal{W}_{\boldsymbol{A}}$ is written out in Theorem 5.1 as a matrix valued differential operator acting on the image $\mu_{\boldsymbol{A}} = \nu \circ W_{\boldsymbol{A}}^{-1}$ of the uniform probability measure ν on $S(\mathbb{C}^N)$ by the numerical range map $W_{\boldsymbol{A}}$. An alternative representation of the Weyl calculus $\mathcal{W}_{\boldsymbol{A}}$ is based on formulae of Herglotz-Petrovsky-Leray [11] for the fundamental solution of the symmetric hyperbolic system (1.4), but the image measure $\mu_{\boldsymbol{A}}$ is not a feature of this representation.

An explicit calculation of the joint spectrum $\gamma(\boldsymbol{A})$ of a pair $\boldsymbol{A} = (A_1, A_2)$ of hermitian matrices is made in Section 5.3, following the approach of [56]. If the matrices A_1 and A_2 commute with each other, then $\gamma(\boldsymbol{A})$ can be identified with the finite set of eigenvalues of the normal matrix $A_1 + iA_2$, otherwise $\gamma(\boldsymbol{A})$