Lecture Notes in Mathematics
1655

J. Azéma M. Emery M. Yor (Eds.)

Séminaire de Probabilités XXXI

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This volume is dedicated to David Williams, whose infectious enthusiasm needed no tunnel to spread across the Channel long before cow madness.

J. Azéma, M. Émery, M. Yor.

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SÉMINAIRE DE PROBABILITÉS XXXI

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Branching processes, the Ray-Knight theorem, and sticky Brownian motion

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1 Introduction

Diffusions with boundary conditions were studied by Ikeda and Watanabe [5] by means of associated stochastic differential equations. Here we are interested in a fundamental example. Let θ and x be real constants satisfying $0 < \theta < \infty$ and $0 \leq x < \infty$. Suppose $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, and that $(X_t; t \geq 0)$ is a continuous, adapted process taking values in $[0, \infty)$ which satisfies the stochastic differential equation

$$X_t = x + \theta \int_0^t I_{\{X_s=0\}} ds + \int_0^t I_{\{X_s>0\}} dW_s, \tag{1.1}$$

where $(W_t; t \geq 0)$ is a real valued $(\mathcal{F}_t)$-Brownian motion. We say that X_t is sticky Brownian motion with parameter θ, started from x. Sticky Brownian motion has a long history. Arising in the work of Feller [3] on the general strong Markov process on $[0, \infty)$ that behaves like Brownian motion away from 0, it has been considered more recently by several authors, see Yamada [12] and Harrison and Lemoine [4], as the limit of storage processes, and by Amir [1] as the limit of random walks.

Ikeda and Watanabe show that (1.1) admits a weak solution and enjoys the uniqueness-in-law property. In [2], Chitashvili shows that, indeed, the joint law of X and W is unique (modulo the initial value of W), and that X is not measurable with respect to W, so verifying a conjecture of Skorokhod that (1.1) does not have a strong solution. The filtration $(\mathcal{F}_t)$ cannot be the (augmented) natural filtration of W and the process X contains some 'extra randomness'. It is our purpose to identify this extra randomness in terms of killing in a branching process. To this end we will study the squared Bessel process, which can be thought of as a continuous-state branching process, and a simple decomposition of it induced by introducing a killing term. We will then be able to realise this decomposition in terms of the local-time processes of X and W. Finally we will prove the following result which essentially determines the conditional law of sticky Brownian motion given the driving Wiener process.

Theorem 1. *Suppose that X is sticky Brownian motion starting from zero, and that W is the driving Wiener process, also starting from zero. Letting $L_t = \sup_{s\leq t}(-W_s)$, the conditional law of X given W satisfies*

$$\mathbb{P}\big(X_t \leq x | \sigma(W)\big) = \exp\big(-2\theta(W_t + L_t - x)\big) \qquad a.s.$$

for $x \in [0, W_t + L_t]$.

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Note in particular that $X_t \in [0, W_t + L_t]$ a.s.. The proof of this result is given in Section 4, and depends on the construction of the pair (X, W) discussed in Section 3. Section 2 is essentially independent, but helps provide us with the intuitive reason for believing Theorem 1.

We begin with a simple but illuminating lemma on sticky Brownian motion, and fix some notation we will need in the sequel.

We denote:

$$A_t^+ = \int_0^t I_{\{X_s>0\}} ds; \qquad \alpha_t^+ = \inf\{u : A_u^+ > t\}; \tag{1.2}$$

$$A_t^0 = \int_0^t I_{\{X_s=0\}} ds; \qquad \alpha_t^0 = \inf\{u : A_u^0 > t\}. \tag{1.3}$$

Then we have

Lemma 2. *If we time change both sides of (1.1) with α^+, the right-continuous inverse of A^+, we find that $(X_{\alpha_t^+}, t \geq 0)$ solves Skorokhod's reflection equation*

$$X_{\alpha_t^+} = W_t^+ + L_t^+.$$

where $W_t^+ = x + \int_0^{\alpha_t^+} I_{\{X_s>0\}} dW_s$ is a Brownian motion, and $L_t^+ = \sup_{s\leq t} \left((-W_s^+) \vee 0\right)$.

Proof. On time changing we have

$$X_{\alpha_t^+} = W_t^+ + \theta A^0_{\alpha_t^+}.$$

Observe that W^+ is a Brownian motion by Lévy's characterization. Now, A_t^0 is a continuous and increasing function of t, A_t^+ is a continuous and strictly increasing function of t, and so $L_t^+ = \theta A^0_{\alpha_t^+}$ is also a continuous and increasing function of t. Furthermore it is constant on the set $\{t : X_{\alpha_t^+} > 0\}$. The criteria of Skorokhod's lemma, see [9], are thus satisfied and $L_t^+ = \sup_{s\leq t}(-W_s^+)$ as claimed. □

This lemma shows us that sticky Brownian motion is just the time change of a reflecting Brownian motion so that the process is held momentarily each time it visits the origin. In this way it spends a real amount of time at the origin, proportional to the amount of local time the reflecting Brownian motion has spent there, in fact,

$$\theta A^0_{\alpha_t^+} = L_t^+. \tag{1.4}$$

The laws of A_t^+,A_t^0 and other quantities can be obtained directly from this, as has been accomplished by Chitashvili and Yor [13].

2 A decomposition of the squared Bessel process

We consider two processes $(R_t, t \geq 0)$ and $(Y_t, t \geq 0)$ satisfying

$$dR_t = 2\sqrt{R_t}\, dB_t - 2\theta R_t\, dt, \qquad R_0 = x, \tag{2.1a}$$

$$dY_t = 2\sqrt{Y_t}\, d\tilde{B}_t + 2\theta R_t\, dt, \qquad Y_0 = 0, \tag{2.1b}$$

where B and $\tilde{B}$ are independent Brownian motions.

Proposition 3. *$V_t = R_t + Y_t$ is a squared Bessel process of dimension 0 started from x.*

Proof. One need only make a simple application of Pythagoras's theorem, following Shiga and Watanabe [11]. We sum the two equations of (2.1) and note that

$$\int_0^t \frac{\sqrt{R_s}\, dB_s + \sqrt{Y_s}\, d\tilde{B}_s}{\sqrt{R_s + Y_s}}$$

is a Brownian motion. □

This simple decomposition can be thought of in the following manner. V_t is the total-mass process of a continuous-state critical branching process and R_t that of a subcritical process. But a subcritical process can be obtained from a critical process by introducing killing at some fixed rate into the latter. Y_t represents the mass of that part of the critical process descended from killed particles. The idea that 'R_t is V_t with killing at rate 2θ' will pervade this paper.

V_t has some finite extinction time $\tau = \inf\{t : V_t = 0\}$, see for example Revuz and Yor [9], and the same is true of R_t, its extinction time being denoted by σ. It is clear that $\tau \geq \sigma$; perhaps surprisingly τ can equal σ, and we will calculate the probability of this. This will be accomplished first via the Lévy-Khintchine formula and then extended using martingale techniques.

Lemma 4. *The laws of the extinction times τ and σ are given by*

$$\mathbb{P}(\tau \in dt) = \frac{x}{2t^2} \exp(-x/2t)\, dt,$$

and

$$\mathbb{P}(\sigma \in dt) = \tfrac{1}{2}x \left[\frac{\theta}{\sinh(t\theta)}\right]^2 \exp\left[\tfrac{1}{2}x\theta\big(1 - \coth(t\theta)\big)\right] dt.$$

Proof. From Pitman and Yor [8],

$$\mathbb{P}(V_t = 0) = \exp(-x/2t),$$

and

$$\mathbb{P}(R_t = 0) = \lim_{\lambda \to \infty} \mathbb{E} \exp(-\lambda R_t) = \exp\left[\tfrac{1}{2}x\theta\big(1 - \coth(t\theta)\big)\right].$$

The lemma follows on differentiating. □

We wish to prove the following.

Proposition 5. *The conditional law of the extinction time of the subcritical process given the extinction time of the critical process satisfies*

$$\mathbb{P}(\sigma = \tau | \tau) = \exp(-2\theta\tau) \qquad a.s..$$

This can be loosely interpreted as the probability that the last surviving particle of the critical process also belongs to the subcritical process, an event that depends on whether there has been any killing along its line of ancestry.

Let us denote the law of a process satisfying

$$dZ_t = 2\sqrt{Z_t}\,dB_t + 2(\beta Z_t + \delta)\,dt, \qquad Z_0 = y,$$

by ${}^{\beta}\mathbb{Q}^{\delta}_{y}$, and the law of the Z-process conditioned to be at x at time t by ${}^{\beta}\mathbb{Q}^{\delta,t}_{y\to x}$. Now the following Lévy-Khintchine formula comes from Yor [14],

$$\mathbb{E}[\exp(-\lambda Y_t)|\sigma(R)] = \exp\left\{-\int n^+(d\epsilon)\int_0^t ds\, 2\theta R_s\Big(1-\exp\big(-\lambda l_{t-s}(\epsilon)\big)\Big)\right\},$$

where n^+ is the restriction of Itô excursion measure for Brownian motion to positive excursions and $l_t(\epsilon)$ the local time at height t of the excursion ϵ. Letting $\lambda \uparrow \infty$, we have

$$\exp\big(-\lambda l_{t-s}(\epsilon)\big) \to \begin{cases} 0 & \text{if } \sup\epsilon > t-s, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, since $n^+(\sup\epsilon > t-s) = 1/2(t-s)$ we obtain

$$\mathbb{P}\big(Y_t = 0|\sigma(R)\big) = \exp\left\{-\int_0^t ds\,\theta R_s/(t-s)\right\}. \tag{2.2}$$

From this it follows that

$$\mathbb{P}(Y_t = 0|\sigma = t) = {}^{-\theta}\mathbb{Q}^{4,t}_{x\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\}. \tag{2.3}$$

Note that, because we are conditioning to hit 0 at time t and not before, we obtain ${}^{-\theta}\mathbb{Q}^{4,t}_{x\to 0}$, and not ${}^{-\theta}\mathbb{Q}^{0,t}_{x\to 0}$ as one might expect, see [8] for a full discussion. To evaluate this we begin by observing that by the change of measure given in Pitman and Yor [8],

$$(2.4)\quad {}^{-\theta}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\} = \frac{{}^{0}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds - \frac{1}{2}\theta^2\int_0^t Z_s\,ds\right\}}{{}^{0}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\frac{1}{2}\theta^2\int_0^t Z_s\,ds\right\}}.$$

Now from [9], under ${}^{0}\mathbb{Q}^{4,t}_{0\to 0}$, Z_t solves, for $u \le t$,

$$Z_u = 2\int_0^u \sqrt{Z_s}\,dB_s + 2\int_0^u [2 - Z_s/(t-s)]\,ds,$$

where B is a Brownian motion. Hence,

$$\theta\int_0^t Z_s/(t-s)\,ds = 2t\theta + \theta\int_0^t \sqrt{Z_s}\,dB_s,$$

but, of course, $\int_0^u \sqrt{Z_s}\,dB_s$ is a martingale with quadratic variation $\int_0^u Z_s\,ds$, so

$$\exp\left\{-\theta\int_0^u \sqrt{Z_s}\,dB_s - \tfrac{1}{2}\theta^2\int_0^u Z_s\,ds\right\}$$

is a martingale too (it's bounded above by $\exp(2\theta t)$!!). We take expectations and have succeeded in evaluating the numerator of (2.4),

$$(2.5)\qquad {}^{0}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds - \tfrac{1}{2}\theta^2\int_0^t Z_s\,ds\right\} = \exp(-2t\theta).$$

We find directly from Pitman and Yor [8] that the denominator satisfies

$$(2.6)\qquad {}^{0}\mathbb{Q}^{4,t}_{0\to 0}\exp\left\{-\tfrac{1}{2}\theta^2\int_0^t Z_s\,ds\right\} = \left[\frac{t\theta}{\sinh(t\theta)}\right]^2.$$

Next we observe, recalling (2.2),

$$(2.7)\qquad \begin{aligned} {}^{-\theta}\mathbb{Q}^{0,t}_{x\to 0}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\} &= \frac{{}^{-\theta}\mathbb{Q}^{0}_{x}\exp\left\{-\theta\int_0^t Z_s/(t-s)\,ds\right\}}{{}^{-\theta}\mathbb{Q}^{0}_{x}I_{\{Z_t=0\}}} \\ &= \frac{{}^{0}\mathbb{Q}^{0}_{x}I_{\{Z_t=0\}}}{{}^{-\theta}\mathbb{Q}^{0}_{x}I_{\{Z_t=0\}}} = \frac{\mathbb{P}(\tau\ge t)}{\mathbb{P}(\sigma\ge t)}. \end{aligned}$$

We can now proceed to

Proof of proposition 5. The Pitman-Yor decomposition, [8],

$${}^{-\theta}\mathbb{Q}^{4,t}_{x\to 0} = {}^{-\theta}\mathbb{Q}^{4,t}_{0\to 0}\oplus {}^{-\theta}\mathbb{Q}^{0,t}_{x\to 0},$$

allows us, combining (2.5),(2.6) and (2.7), to compute $\mathbb{P}(\tau = t|\sigma = t)$. Then we have

$$\mathbb{P}(\sigma = t|\tau = t) = \mathbb{P}(\tau = t|\sigma = t)\frac{\mathbb{P}(\sigma\in dt)}{\mathbb{P}(\tau\in dt)},$$

and substituting from the lemma we are done. □

We will now extend this result by conditioning on the whole of V, instead of just its extinction time. We will need the following lemma, which is perhaps of some independent interest.

Lemma 6. *Suppose M and N are continuous, orthogonal martingales with respect to a filtration $(\mathcal{F}_t; t\ge 0)$, and suppose that M has the following representation property. Any bounded, $\sigma(M)$-measurable variable Φ is of the form*

$$\Phi = c + \int_0^\infty H_t\,dM_t,$$

where H_t is $\mathcal{F}_t$-previsible, and $c\in\sigma(M_0)$. Let $\mathcal{G}_t = \mathcal{F}_t\vee\sigma(M)$, then N is a $\mathcal{G}_t$-martingale.

Proof. By an application of the monotone-class lemma, it suffices to show that for bounded $\sigma(M)$-measurable variables Φ,

$$\mathbb{E}[\Phi(N_t - N_s)|\mathcal{F}_s] = 0.$$

But, by the representation property,

$$\begin{aligned}\mathbb{E}[\Phi(N_t - N_s)|\mathcal{F}_s] &= \mathbb{E}\left[\left\{\int_0^t H_u\, dM_u\right\}(N_t - N_s)\middle|\mathcal{F}_s\right] \\ &= \mathbb{E}[(H\cdot M)_t N_t - (H\cdot M)_s N_s|\mathcal{F}_s] \\ &= 0,\end{aligned}$$

since $(H\cdot M)$ and N are orthogonal. □

Now on the stochastic interval $[0,\tau)$ we define

$$\Theta_t = \frac{R_t}{V_t}\exp(2\theta t).$$

Applying Itô's formula gives

$$d\Theta_t = \left\{2\frac{\sqrt{R_t}}{V_t}\, dB_t - \frac{R_t}{V_t^2}\, dV_t\right\}\exp(2\theta t),$$

which shows Θ_t to be a local martingale on $[0,\tau)$. Moreover, since $\Theta_t < \exp(2\theta t)$, Θ_t tends to a finite limit as $t\uparrow\tau$, and if we define $\Theta_t = \Theta_{\tau-}$ for $t \geq \tau$, then Θ_t is a martingale for $0 \leq t < \infty$.

If we continue to calculate with Itô's formula, we find that, for $t < \tau$,

(2.8a) $$d\Theta_t dV_t = 0$$

(2.8b) $$d\Theta_t d\Theta_t = 4\frac{\Theta_t}{V_t}\big(\exp(2\theta t) - \Theta_t\big).$$

Thus we have proved

Lemma 7. *Θ_t is a $\mathcal{F}_t$-martingale with quadratic variation*

$$[\Theta]_t = \int_0^{t\wedge\tau} ds\, 4\frac{\Theta_s}{V_s}\big(\exp(2\theta s) - \Theta_s\big),$$

and furthermore Θ is orthogonal to V.

So if we put $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V)$, we can apply Lemma 6 to deduce that Θ_t is a $\mathcal{G}_t$-martingale. Moreover, τ is $\mathcal{G}_0$- measurable, and so for any positive constant K,

$$\mathbb{E}\Theta_\infty I_{\{\tau<K\}} = \mathbb{E}\Theta_0 I_{\{\tau<K\}},$$

since $\Theta_t I_{\{\tau<K\}}$ is a bounded $\mathcal{G}_t$-martingale. But as $K\uparrow\infty$ we obtain

$$\mathbb{E}\Theta_\infty = \mathbb{E}\Theta_0 = 1,$$

whence Θ is uniformly integrable. Now we are able to prove

Proposition 8. *The conditional law of the extinction time of the subcritical process given* $\sigma(V_t; 0 \leq t < \infty)$ *satisfies*

$$\mathbb{P}\big(\sigma = \tau | \sigma(V)\big) = \exp(-2\theta\tau) \qquad a.s..$$

Proof. We have already remarked that $\Theta_{\tau-}$ exists and hence $[\Theta]_\tau$ is finite almost surely. It is easy to confirm, for example by time inversion, that $\int_0^\tau V_s^{-1}\, ds = \infty$, and thus we deduce from the formula for the quadratic variation process of Θ, given in Lemma 7, that $\Theta_s(\exp(2\theta s) - \Theta_s) \to 0$ as $s \uparrow \tau$. Hence Θ_τ is either 0 or $\exp(2\theta\tau)$. Furthermore,

$$\mathbb{E}[\Theta_\tau | \mathcal{G}_0] = \mathbb{E}[\Theta_0 | \mathcal{G}_0] = 1,$$

and so,

$$\mathbb{P}\big(\Theta_\tau = \exp(2\theta\tau) | \sigma(V)\big) = \exp(-2\theta\tau).$$

Now observe that $\tau > \sigma$ implies that $\Theta_\tau = 0$ (but the converse isn't so evident!), whence

$$\mathbb{P}\big(\tau = \sigma | \sigma(V)\big) \geq \exp(-2\theta\tau).$$

But

$$\mathbb{P}(\tau = \sigma | \tau) = \mathbb{E}\big[\mathbb{P}\big(\tau = \sigma | \sigma(V)\big) \big| \tau\big] = \exp(-2\theta\tau),$$

implying the desired equality. □

3 A decomposition of Brownian motion

It is now well known, as excellently described by Le Gall [7], that if we interpret the squared Bessel process of dimension zero as a continuous-state branching process then the associated genealogical structure is carried by Brownian excursions. In this section we will give a decomposition of Brownian motion that corresponds to the decomposition of the squared Bessel process induced by the killing considered previously. By looking at local times we will be able to recover Proposition 3.

To begin we recall:

Theorem 9 (Ray-Knight). *If* $\bar{W}_t$ *is reflecting Brownian motion, starting from zero, with* l_t^y *its local time at level* y*, then, letting* $\tau_x = \inf\{t : l_t^0 \geq x\}$*, we have* $(l_{\tau_x}^y, y \geq 0)$ *is a squared Bessel process of dimension 0 started from* x*.*

If we introduce drift we can obtain the subcritical process of the previous section in a similar manner.

Theorem 10. *If* $\bar{S}_t$ *is reflecting Brownian motion with drift* θ *towards the origin, starting from zero, and if* l_t^y *is its local time at level* y*, then letting* $\tau_x = \inf\{t : l_t^0 \geq x\}$*, we have the law of the process* $(l_{\tau_x}^y, y \geq 0)$ *is* ${}^{-\theta}\mathbb{Q}_x^0$*.*