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ELEMENTS OF GAME THEORY

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This book treats in a popular manner the elements of game theory and some methods for solying matrix games. It contains almost no proofs and illustrates the basic principles with examples. To be able to read the book, an acquaintance with the elements of probability theory and calculus is enough. The book is intended to disseminate the ideas of game theory which have practical economic or military applications.

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ИЗДАТЕЛЬСТВО «МИР»

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Ye. S. Venttsel

ELEMENTS OF GAME THEORY

Translated from the Russian
by
Vladimir Shokurov

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Е. С. Вентцель

ЭЛЕМЕНТЫ ТЕОРИИ ИГР

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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Contents

1. The Subject-matter of Game Theory. Basic Concepts

When solving practical (economic, military or other) problems one often has to analyse situations in which there are two, or more quarrelling parties pursuing conflicting objectives, and where the outcome of each action of one party depends on the opponent's choice of a course of action. Such situations will be called "conflict situations".

One can cite numerous examples of conflict situations from various practical situations. All situations arising in the course of military action are conflict situations: each of the contending parties takes every available measure to prevent the opponent from succeeding. Conflict situations also arise in choosing a weapon or a mode of its combat use, and in general, in planning military operations; every decision in this area must be made assuming that the opponent's action will be the least favourable one. A number of economic situations (especially those where there is free competition) are conflict situations. The contending parties here are firms, industrial enterprises, etc.

The need to analyse situations of this kind has brought about the development of special mathematical methods. The theory of games is in fact a mathematical theory of conflict situations. The aim of the theory is to elaborate recommendations for each of the opponents to act rationally in the course of a conflict situation.

All conflict situations that occur in practice are very complicated and their analysis is hampered by many attendant factors. For a mathematical analysis of a situation to be possible, it is necessary to disengage oneself from these secondary factors and construct a simplified, formalized model of the situation. Such a model will be called a "game".

A game differs from a real conflict situation in that it is played according to definite *rules*. Man has long used such formalized models of conflict situations – *games* in the literal sense of the word, for example chess, checkers, card games and so on. Each of these games takes the form of a contest proceeding according to certain rules and ending in one or another player's "victory" (or gain).

Such formally regulated, artificially arranged games provide the most suitable material for illustrating and learning the basic concepts of game theory. The terminology borrowed from such games is used in the analysis of other conflict situations as well;

the convention has been adopted of referring to the parties taking part in them as "players" and to the outcome of an encounter as a party's "gain" or "payoff".

A game may be clash of interests of two or more opponents; in the first case the game is called a two-person game and in the second it is called a multiperson game. The participants of a multiperson game may, in the course of the game, form coalitions, constant or temporary. If there are two constant coalitions in a multiperson game, it becomes a two-person game. In practice the most important games are two-person games. We shall confine ourselves to these games only.

We shall begin our presentation of elementary game theory by formulating some basic concepts. We shall consider the two-person game in which two players, A and B, with opposing interests take part. By a "game" we shall mean an arrangement consisting of a number of actions taken by parties A and B. For a game to be treated mathematically it is necessary to formulate the *rules of the game* exactly. The "rules of a game" are understood to be a set of conditions which regulate the conceivable alternatives for each party's course of action, the amount of information each party has about the other party's behaviour, the sequence of alternating the "moves" (individual decisions made in the course of the game), and the *result* or *outcome* to which the given totality of moves leads. This result (gain or loss) does not always have a quantitative expression, but it is usually possible to express the result by a certain number by establishing some measuring scale. For example, it might be agreed in chess to assign the value $+1$ to a victory, the value -1 to a defeat, and the value 0 to a draw.

A game is called a *zero-sum* game if one player gains what the other loses, i. e. if the sum of both parties' gains is equal to zero. In a zero-sum game the players' interests are completely opposed. We shall consider only zero-sum games in the following.

Since in a zero-sum game one player's gain is the other player's gain taken with the opposite sign, it is obvious that in analysing such a game it is possible to consider just one player's gain. Let us take player A, for example. For the sake of convenience, in what follows we shall arbitrarily refer to party A as "we" and to party B as "the opponent".

Party A ("we") will always be regarded as the winner and party B ("the opponent") as the loser. Evidently this formal condition does not imply any real advantage to the first player. It can easily be seen that it can be replaced by the opposite condition if the sign of the gain is reversed.

We shall consider the development of a game in time as a series of successive steps or "moves". In game theory a *move* is a choice of an alternative from the alternatives that are allowed by the rules of the game. Moves can be classified as *personal* or *chance* moves.

A *personal move* is a player's deliberate choice of one of the moves possible in the given situation, and its realization.

An example of a personal move is a move in a game of chess. In making his move, a player makes a deliberate choice among the alternatives possible for a given disposition of pieces on the chessboard.

The set of possible alternatives is stipulated for each personal move by the rules of the game and depends on the totality of both parties' previous moves.

A *chance move* is a choice among a number of possibilities which is realized not by the player's decision but by some random device (the tossing of a coin or a dice, the shuffling and dealing of cards, etc.). For example, dealing the first card to a bridge player is a chance move with 52 equally possible alternatives.

For a game to be mathematically definite, the rules must indicate for each chance move the *probability distribution* of the possible outcomes.

Some games may contain only chance moves (the so-called games of pure chance) or only personal moves (chess, checkers). Most card games are of mixed type, i. e. they consist of both chance and personal moves.

Games are classified not only according to the nature of the moves (into personal and chance moves), but also according to the nature and amount of information available to either player concerning the other's actions. A special class of games is formed by "games with perfect information". A *game with perfect information* is a game in which either player knows at each move the results of all the previous moves, both personal and chance. Examples of games with perfect information are chess, checkers, and the well-known game of "noughts-and-crosses".

Most of the games of practical importance are not games with perfect information since the lack of information about the opponent's actions is usually an essential element of conflict situations.

One of the basic concepts of game theory is the concept of a "strategy".

A *strategy* for player is a set of rules unambiguously determining the choice of every personal move of the player, depending on the situation that has arisen in the course of the game.

The concept of strategy should be explained in more detail.

A decision (choice) for each personal move is usually made by the player in the course of the game itself depending on the particular situation that has arisen. Theoretically, the situation will not be altered, however, if we imagine that all the decisions are made by the player *in advance*. To that end the player would have to make a list of all situations that might occur in the course of the game beforehand and foresee his decision for each of them. This is possible in principle (if not in practice) for any game. If such a system of decisions is adopted, it means that the player has chosen a definite *strategy*.

A player who has chosen a strategy may now abstain from taking part in the game personally and substitute for his participation a list of rules to be applied for him by some disinterested person (a referee). The strategy may also be given to an automatic machine in the form of a certain programme. It is in this way that modern electronic computers play chess.

For the concept of "strategy" to have sense, a game must have personal moves; there are no strategies in games comprising only chance moves.

Games are classified into "finite" and "infinite" ones depending on the number of possible strategies.

A game is said to be *finite* if either player has only a finite number of strategies.

A finite game in which player A has m strategies and player B n strategies is called an $m \times n$ game.

Consider an $m \times n$ game between two players, A and B ("we" and "our opponent").

We denote our strategies by $A_1, A_2, \ldots, A_m$, and our opponent's strategies by $B_1, B_2, \ldots, B_n$.

Suppose that either party has chosen a definite strategy; let it be A_i in our case, and B_j in our opponent's.

If the game contains only personal moves, then the choice of strategies A_i, B_j unambiguously determines the outcome of the game – our gain (payoff). We denote it by a_{ij}.

If the game consists of chance moves as well as personal ones, then the gain for a pair of strategies A_i and B_j is a random quantity depending on the outcomes of all chance moves. In this case the natural estimate of the gain expected is its *mean value* (mathematical expectation). We shall denote by the same symbol a_{ij} both the gain (payoff) itself (in games without chance moves) and its mean value (in games with chance moves).

Suppose we know the value a_{ij} of the gain or payoff (or the average values) for each pair of strategies. The values a_{ij} can be written in the form of a rectangular array (or matrix) the rows of which correspond to our strategies (A_i) and the columns to our opponent's strategies (B_j). This array is called a *gain* or *payoff* matrix, or simply the *matrix of the game* or *game matrix*.

The matrix of an $m \times n$ game is of the form:

A \ B	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
⋮	⋮	⋮	⋮	⋮
A_m	a_{m1}	a_{m2}	...	a_{mn}

We shall briefly denote a game matrix by $\|a_{ij}\|$.

Consider a few elementary examples of games.

EXAMPLE 1. Two players, A and B, without looking at each other, each places a coin face up on a table showing either heads or tails as they like. If both choose the same side (either heads or tails), then player A takes both coins; otherwise they are taken by player B. Analyse the game and construct its matrix.

SOLUTION. The game contains only two moves, our move and our opponent's move; both are personal. It is not a game with perfect information since at the moment a move is made the player who makes it does not know what the other player will do.

Since either player has only one personal move, a player's strategy is a choice for this single personal move.

There are two strategies for us: choosing heads, A_1, and choosing tails, A_2; there are the same two strategies for our opponent: heads, B_1, and tails, B_2. So this is a 2×2 game. Let a gain of the coin count $+1$. The game matrix is given below.

This game, simple as it is, may help us to understand some essential ideas of game theory.

First assume that the game is played only once. Then it is evidently useless to speak of any "strategies" for the players

B / A	B_1 (heads)	B_2 (tails)
A_1 (heads)	1	−1
A_2 (tails)	−1	1

being more clever than the others. Either player may equally reasonably make either decision. When the game is repeated, however, the situation changes.

Indeed, assume that we (player A) have chosen some strategy (say A_1) and are keeping to it. Then from our initial moves our opponent will guess what our strategy is and respond to it in the manner least advantageous to us, i. e. he will choose tails. It is clearly not advantageous for us to always play only one of our strategies; in order not to lose we must sometimes choose heads and sometimes tails. However, if we alternate heads and tails in any definite order (for example, one after the other), our opponent may also guess this and counter our strategy in the worst manner for us. Evidently a reliable method which would guarantee that our opponent is not aware of our strategy is to arrange our choice of each move in such a way that we do not know it in advance ourselves (this could be ensured by tossing a coin, for instance). Thus, through an intuitive argument we have approached one of the essential concepts of game theory, that of a "mixed strategy", i. e. a strategy in which "pure" strategies – A_1 and A_2 in this case – are alternated randomly with certain frequencies. In this example, it is known in advance from symmetry considerations that strategies A_1 and A_2 should be alternated with the same frequency; in more complicated games, the decision may be far from being trivial.

EXAMPLE 2. Players A and B each write down simultaneously and independently of each other, one of the three numbers: 1, 2, or 3.

If the sum of the numbers they have written down is even, then B pays A that sum in dollars; if the sum is odd, then, on the contrary, A pays that sum to B. Analyse the game and construct its matrix.

SOLUTION. The game consists of two moves, both of which are personal. We (A) have three strategies: writing down 1, A_1; writing down 2, A_2; and writing down 3, A_3. Our opponent (B) has the same three strategies. This is a 3×3 game with the matrix given below.

B \ A	B_1	B_2	B_3
A_1	2	−3	4
A_2	−3	4	−5
A_3	4	−5	6

Evidently our opponent can, as in the previous case, respond to any strategy chosen in the way which is worst for us. Indeed, if we choose strategy A_1, for instance, our opponent will always counter it with strategy B_2; strategy A_2 will always be countered with strategy B_3 and strategy A_3 with strategy B_2; thus any choice of a definite strategy will inevitably lead us to a loss.* The solution of this game (i. e. the set of the most advantageous strategies for both players) is given in Section 5.

EXAMPLE 3. We have three kinds of weapon at our disposal, A_1, A_2, and A_3; the enemy has three kinds of aircraft, B_1, B_2, and B_3. Our goal is to hit an aircraft, while the enemy's goal is to keep it unhit. When armament A_1 is used, aircraft B_1, B_2, and B_3 are hit with probabilities 0.9, 0.4, and 0.2, respectively; when armament A_2 is used, they are hit with probabilities 0.3, 0.6, and 0.8; and when armament A_3 is used, with probabilities 0.5, 0.7 and 0.2. Formulate the game in terms of game theory.

SOLUTION. The situation can be regarded as a 3×3 game with two personal moves and one chance move. Our personal move is to choose a kind of a weapon; the enemy's personal move is to choose an aircraft to take part in the combat. The chance move is to select the weapon; this move may or may not end with hitting the aircraft. Our gain is unity if the aircraft is hit, otherwise it is zero. Our strategies are the three alternative weapons; the enemy's strategies are the three alternative aircraft.

The mean value of the gain in each of the specified pairs of strategies is just the probability of hitting a given aircraft with a given weapon. The game matrix is given below.

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* One should not forget, however, that our opponent's position is as bad as ours.

B / A	B_1	B_2	B_3
A_1	0.9	0.4	0.2
A_2	0.3	0.6	0.8
A_3	0.5	0.7	0.2

The aim of game theory is to work out recommendations for the player' rational behaviour in conflict situations, i. e. to determine an "optimal strategy" for each player.

In game theory, an *optimal strategy* for a player is a strategy which, when repeated many times, assures him the maximum possible average gain (or, which amounts to the same thing, the minimum possible average loss). The argument for the choice of this strategy is based on the assumption that the opponent is at least as rational as we ourselves are and does everything to prevent us from achieving our object.

All recommendations in game theory are deduced from these principles. Consequently, no account is taken of risk elements, which are inevitably present in every real strategy, nor of possible miscalculations or errors made by the players.

Game theory, like any mathematical model of a complex phenomenon, has its limitations. The most serious limitation is the fact that the gain is artificially reduced to only one number. In most practical conflict situations, when working out a rational strategy one has to take into account several criteria of successful action, i. e. several numerical parameters rather than one. A strategy optimal according to one criterion is not necessarily optimal according to another. However, by realizing these limitations and therefore not blindly following the recommendations obtained by game theoretic methods, one can still employ mathematical game theory techniques to work out a strategy which would at any rate be "acceptable", if not "optimal".

2. The Lower and the Upper Value of the Game. The "Minimax" Principle

Consider an $m \times n$ game having the following matrix.

We shall denote by the letter i the number of our strategy and by the letter j the number of our opponent's strategy.

B / A	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}		a_{1n}
A_2	a_{21}	a_{22}		a_{2n}
. . .	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}

We undertake to determine our optimal strategy. We shall analyse each of our strategies sequentially starting with A_1. Choosing strategy A_i we must always count on our opponent responding to it with that strategy B_j for which our gain a_{ij} is minimal. We shall determine this value of the gain, i. e. the smallest of the numbers a_{ij} in the ith row. We shall denote it by α_i:

$$\alpha_i = \min_j a_{ij} \tag{2.1}$$

Here the symbol $\min_j$ (minimum over j) denotes the minimum value of the given parameter for all possible j.

We shall write the numbers α_i next to the matrix above in an additional column.

By choosing some strategy A_i we can count on winning, as a result of our opponent's rational actions, not more than α_i. It is natural that, by acting in the most cautious way and assuming the most rational opponent (i. e. avoiding any risk), we must decide on a strategy A_i for which number α_i is maximal. We shall denote this maximal value by α:

$$\alpha = \max_i \alpha_i$$

or, taking into account formula (2.1)

$$\alpha = \max_i \min_j \alpha_{ij}$$

The quantity α is called the *lower value of the game*, or else the *maximin gain* or simply the *maximin.*

A \ B	B_1	B_2	…	B_n	α_i
A_1	a_{11}	a_{12}	…	a_{1n}	α_1
A_2	a_{21}	a_{22}	…	a_{2n}	α_2
⋮	⋮	⋮	⋮	⋮	⋮
A_m	a_{m1}	a_{m2}	…	a_{mn}	α_m
β_j	β_1	β_2	…	β_m	

The number α lies in some row of the matrix; player A's strategy which corresponds to this row is called a *maximin strategy*.

Evidently, if we keep to a maximin strategy, then, whatever our opponent's behaviour, we are *assured a gain which is in any case not less than* α. Because of this the quantity α is called the "lower value of the game". This is the guaranteed minimum which we can assure ourselves by keeping to the most cautious ("play safe") strategy.

Evidently, a similar argument can be carried out for opponent B. Since the opponent is interested in minimizing our gain, he must examine each of his strategies in terms of the maximum gain for it. Therefore we shall write out the maximal values a_{ij} for each column at the bottom of the matrix:

$$\beta_j = \max_i a_{ij}$$

and find the minimal value β_j:

$$\beta = \min_j \beta_j$$

or

$$\beta = \min_j \max_i a_{ij}$$

The quantity β is called the *upper value of the game*, or else the "minimax". The opponent's strategy corresponding to the minimax gain is called his "minimax strategy".