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Edited by A. Dold and B. Eckmann

Subseries: Department of Mathematics, University of Maryland
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J.C. Alexander (Ed.)

Dynamical Systems

Proceedings, University of Maryland 1986–87

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Dynamical Systems

Proceedings of the Special Year
held at the University of Maryland, College Park, 1986–87

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Douglas C. McMahon, 1947-1986

Professor Douglas McMahon of Arizona State University died December 29 in a mountain climbing accident on Mount Orizaba in Mexico.

Doug, who participated in the ergodic theory conference at Maryland in October, was an active researcher in topological dynamics. He received his Ph.D. in 1972 from Case-Western Reserve University under the direction of Ta Sun Wu, with whom he continued to collaborate. Among the topics he studied were local almost periodicity, weak mixing and its generalizations, structure theorems (including a non-metric Furstenberg structure theorem for distal flows), disjointness, and the equicontinuous structure relation. He also contributed a number of illuminating examples. In particular, we single out two of his results. The first (Trans. Amer. Math. Soc. 236 (1978), 225-237) is an elegant proof that in a minimal flow which admits an invariant measure, the regionally proximal relation is an equivalence relation (and hence coincides with the equicontinuous structure relation). The second (Proc. Amer. Math. Soc. 98 (1986), 175-179) is an ingenious proof of a "multiple disjointness" theorem: Let T be an abelian group and let (X,T) be a family of weakly mixing regular minimal flows which are pairwise disjoint. Then the product flow $(\Pi X_i, T)$ is minimal.

Doug was very much a lover of the out-of-doors, an avid climber, hiker and rafter, who had pursued these activities extensively throughout North America.

To many of the participants of the 1986-87 Special Year in Ergodic Theory and Dynamics at the University of Maryland, Doug was both a friend and stimulating colleague. We shall all miss him.

The Organizers

During the academic year 1986–1987, the Mathematics Department of the University of Maryland devoted its special year to various aspects of Dynamics. In addition to having a number of both long and short-term visitors, the Department sponsored three separate conferences on different aspects of dynamics. These were: *Ergodic Theory and Topological Dynamics* October 13 through October 17, 1986, *Symbolic Dynamics and Coding Theory*, December 1 through December 5, 1986, and *Smooth Dynamics, Dynamics and Applied Dynamics*, March 9 through March 13, 1987.

The papers in this proceedings reflect the richness and diversity of the subject of dynamics. Some of the papers in this proceedings are lectures given at the conferences, some are work which was in progress during the special year, and some are work which was done because of questions and problems raised at the conferences. In addition, a paper of John Milnor and William Thurston, versions of which have been available as notes, but which has not been published, is included. The editor would like to thank those individuals, both at Maryland and elsewhere, who acted as referees, often on short notice.

One of the reasons for the success of the special year was the financial support of the Department of Mathematics of the University of Maryland, the Institute for Physical Science and Technology of the University of Maryland, and the National Science Foundation (through grant DMS86-10332). In addition, the Department of Mathematics contributed administrative and logistical support, and some typing for this proceedings.

The special year, and especially the conferences, were full of mathematical excitement. The editor and the organizers hope these papers convey some of that excitement.

J. C. Alexander
November, 1987

Conference 1

Michael Bramton
Robert Burton
James Campbell
Joe Christy
Ethan Coven
Andres del Junco
Stanley Eigen
David Ellis
Jacob Feldman
Sebastien Ferenczi
Adam Fieldsteel
Alan Forrest
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G. R. Goodson
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Jim Propp
Maury Rahe
E. Arthur Robinson
Michael Sears
Cesar Silva
John Smillie
Meir Smorodinsky
Jean-Paul Thouvenot
Paul Trow
Selim Tuncel
Dan Ullman

Conference 2

Roy Adler
Jon Ashley
James T. Campbell
Elise Cawley
Ethan Coven
Imre Csizar
Albert Fathi
Leo Flatto

Conference 2 (cont'd.)

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Charles Jacobson
Razmik Karabed
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Marakazu Nasu
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Frank Rhodes
E. Arthur Robinson
Fred Roush
Klaus Schmidt
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Matt Stafford
Paul Trow
Selim Tuncel
Jack Wagoner
James Ward
Susan Williams

Conference 3

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Paul Blanchard
Keith Burns
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Joe Christy
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Ketty de Rezende
Manfred Denker
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David Fried
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M. Gilboa
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John Guckenheimer
Ursula Hamenstadt
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Discerning Fat Baker's Transformations

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1. Introduction.

Define a piecewise-linear transformation $T = T_\beta$ of the square $\{(x,y): 0 \le x, y \le 1\}$ for $\frac{1}{2} \le \beta < 1$:

$$T_\beta(x,y) = \begin{cases} (2x, \beta y), & \text{if } 0 \le x \le \frac{1}{2}, \\ (2x-1, \beta y + (1-\beta)), & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{1.1}$$

For $\beta = \frac{1}{2}$, T_β is the classical baker's transformation. More generally, T_β stretches horizontally by a factor of 2 and compresses vertically by a factor of β. Hence the name 'fat baker's transformation.' For $\beta > \frac{1}{2}$, T_β is not invertible. In [**1**], these transformations were introduced and studied as interesting examples in the context of fractional metric dimensions. Moreover, each T_β has a natural (Bowen-Ruelle) ergodic invariant measure μ_β which can be more-or-less explicitly written down (either combinatorially or in terms of its characteristic function).

Here we consider the T_β in the context of measure-preserving transformations. Let Σ_2 (resp. Σ_2^+) be the two-sided (one-sided) Bernoulli shift on two symbols of equal weight. Of course $T_{\frac{1}{2}}$ is conjugate to Σ_2. More generally, we show the following.

Theorem 1. *T_β is a factor of Σ_2. Thus T_β is a K-endomorphism. The projection of T_β to the x-axis represents Σ_2^+ as a (marginal) factor of T_β. Thus the entropy of T_β equals* $\log 2$ *for all β. The natural extension of T_β is Σ_2.*

We also consider whether, for $\beta \ne \beta'$, T_β is conjugate to $T_{\beta'}$. We do not give a complete answer to the problem. We show that for those β for which μ_β is absolutely continuous with respect to Lebesgue measure, the T_β are different for different β. It is not known which μ_β are absolutely continuous, although it is conjectured to be so for almost all β, $\frac{1}{2} \le \beta < 1$. It is known that μ_β is absolutely continuous for almost all $\beta > \beta_0$ for some $\beta_0 < 1$ [**3**]. In particular, we obtain an uncountable number of distinct T_β. On the other hand, it known there is at least a countable set of β for which μ_β is not absolutely continuous [**2**]. See also [**4,7,13**].

To differentiate between two T_β, of course we seek a measure-theoretic invariant which is different for the two transformations. The most obvious invariant, the entropy, equals log 2 for all the T_β, and hence does not work. Since the T_β are not invertible, some measure of non-invertibility could be considered. One choice is some function of the Jacobian $T'_\beta = d\mu_\beta T_\beta / d\mu_\beta$ of T_β [**11**, chap. 10]. In fact, we prove the following result.

Theorem 2. *Consider β for which μ_β is absolutely continuous with respect to Lebesgue measure. Then*

$$\int \log T'_\beta \, d\mu_\beta = \log(2\beta). \tag{1.2}$$

Also for such β, $\int T'_\beta \, d\mu_\beta$ is a strictly increasing function and $\mu_\beta(\{T'_\beta = 1\})$ is a strictly decreasing function of β.

To compute with Jacobians, one has to have good control over null sets. In particular, one needs to know that the transformation is positively non-singular. Theorem 2 is valid for any β for which

* Partially supported by N.S.F.

T_β is positively non-singular. The following result is what permits us to have such control when μ_β is absolutely continuous with respect to Lebesgue measure.

Theorem 3. *If μ_β is absolutely continuous with respect to Lebesgue measure, then it is equivalent to Lebesgue measure (i.e. has the same null sets).*

2. The measures μ_β

The natural measure for T_β, $\beta \geq \frac{1}{2}$, is the product of the uniform (Lebesgue) measure in the horizontal direction and a measure σ_β in the vertical direction. The measure σ_β is an infinite convolution of (two-point) Bernoulli measures. It can be described combinatorially as follows. For any interval $I \subset [0,1]$, and integer $N \geq 1$, let

$$\sigma_\beta^{(N)}(I) = 2^{-(N+1)}\left[\#\left\{(a_0, a_1, \ldots, a_N) : (1-\beta)\sum_{i=0}^{N} a_i\beta^i \in I, \text{ with each } a_i = 0 \text{ or } 1,\ i = 0, \ldots, N\right\}\right]. \quad (2.1)$$

That is $\sigma_\beta^{(N)}(I)$ is the fraction of β-adic sums $\sum_{i=0}^{N} a_i\beta^i$, with coefficients 0 or 1, of length N contained in the interval I. The measure $\sigma_\beta^{(N)}(I)$ is a finite convolution of Bernoulli measures and

$$\sigma_\beta(I) = \lim_{N\to\infty} \sigma_\beta^{(N)}(I). \quad (2.2)$$

For example, if $\beta = \frac{1}{2}$, then σ_β is uniform. If dx is Lebesgue measure on the interval $[0,1]$, then [**1**]

$$\mu_\beta = dx \times \sigma_\beta. \quad (2.3)$$

on the square $[0,1] \times [0,1]$.

The continuous measures σ_β were defined and studied in the 1930's in the context of harmonic analysis. In particular, the question of which σ_β are absolutely continuous was studied. The complete answer remains open. A number of results are known, of which we quote three:

(i) the limit σ_β exists and is pure, i. e., σ_β is either absolutely continuous or totally singular [**6**],

(ii) there exist β for which σ_β is totally singular (e. g., $\beta = \frac{1}{2}(\sqrt{5}-1)$) [**2**] (more generally for β the reciprocal of a Pisot-Vijayarghavan number: see [**13**]),

(iii) there exists $\beta_0 < 1$ such that σ_β is absolutely continuous for almost all $\beta > \beta_0$ [**3**] (there are other isolated β for which σ_β is known to be absolutely continuous [**4**]).

Proof of Theorem 1.

Let Σ_2 be the shift defined on the sequence space

$$\left\{(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) : \text{ each } a_i = 0 \text{ or } 1\right\}.$$

Define a map F from this sequence space to the square $[0,1] \times [0,1]$:

$$F(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) = \left(\sum_{i<0} a_i 2^i, \sum_{i\geq 0} a_i\beta^i\right).$$

It is routine to verify from the definitions that F *is a semiconjugacy from* Σ_2 *to* T_β; that is, F is measure-preserving and $F\Sigma_2 = T_\beta F$. Hence, T_β is a factor of Σ_2. Since the horizontal component of μ_β is uniform, it is clear that the projection onto the horizontal axis is a semiconjugacy from T_β to Σ_2^+. Since semiconjugacies do not increase entropy, the entropy of T_β equals log 2.

Let $\tilde{T}_\beta$ be the natural extension of T_β. That is, $\tilde{T}_\beta$ is invertible and if S is any invertible measure-preserving map and f is a semiconjugacy from S to T_β, then there is a semiconjugacy $\tilde{f}$ from S to $\tilde{T}_\beta$. In particular, let $S = \Sigma_2$ and $f = F$ above. Then $\tilde{F}$ represents $\tilde{T}_\beta$ as a factor of Σ_2. Hence [**9**] $\tilde{T}_\beta$ is a Bernoulli shift and hence [**8**] is isomorphic to Σ_2.

This completes the proof of Theorem 1 and in particular shows that entropy does not distinguish the T_β.

3. Absolutely continuous μ_β

Proof of Theorem 3.

Let dy be Lebesgue measure on $[0,1]$ and σ_β as in (2.1–2.2). Assuming σ_β is absolutely continuous with respect to dy, let $h(y) = d\sigma_\beta/dy$. Then if $dx\,dy$ is Lebesgue measure on the square $[0,1]\times[0,1]$, the derivative $d\mu_\beta/dx\,dy = h(y)$. To prove Theorem 3, we show: *$h(y) > 0$ almost everywhere* mod $dx\,dy$.

Divide the square into three regions by two horizontal lines, one at $y = 1-\beta$ and one at $y = \beta$. Call the three regions the *lower*, *central* and *upper* regions. Note that the preimage under T_β of the lower (resp. upper) region is contained in the left (right) half of the square, whereas the preimage of the central region intersects both halves. Consider a small rectangle $I \times J$ in the lower region which is the product of intervals I and J of lengths $|I|$ and $|J|$. The preimage T_β^{-1} is a rectangle which is the product of the intervals $I/2$ and J/β. By the invariance of μ_β,

$$\int_{I\times J} h(y)\,dx\,dy = \int_{I/2\times J/\beta} h(y)\,dx\,dy,$$

so that

$$|I|\int_J h(y)\,dy = \frac{|I|}{2}\int_{J/\beta} h(y)\,dy.$$

Dividing by $|J|$ and letting $|J|\to 0$, we obtain the functional relation

$$h(y) = (2\beta)^{-1}h(y/\beta) \text{ almost everywhere mod } dy \text{ for } y < 1-\beta. \tag{3.1}$$

Similarly, considering the upper region,

$$h(y) = (2\beta)^{-1}h\left((y-(1-\beta))/\beta\right) \text{ almost everywhere mod } dy \text{ for } y > \beta. \tag{3.2}$$

For $I\times J$ in the central region,

$$h(y) = (2\beta)^{-1}h(y/\beta) + (2\beta)^{-1}h\left((y-(1-\beta))/\beta\right) \text{ almost everywhere mod } dy \text{ for } 1-\beta < y < \beta. \tag{3.3}$$

Let

$$A = \{y\colon h(y) = 0, 2-\beta^{-1} < y < 1\} \tag{3.4}$$

Define the map Q on the interval $[2-\beta^{-1}, 1]$ by the formula

$$Qy = \begin{cases} y/\beta & \text{if } 2-\beta^{-1} < y < \beta, \\ (y-(1-\beta))/\beta & \text{if } \beta \le y < 1. \end{cases} \tag{3.5}$$

By (3.1)–(3.3), $QA \subset A$.

Consider Q' defined on $[0,1]$ by $Q'y = \beta^{-1}y \bmod 1$. The affine map $g\colon [0,1]\to[2-\beta^{-1},1]$ defined by $g(y') = (1-\beta^{-1})y' + 1$ satisfies $gQ' = Qg$. Let $A' = g^{-1}(A)$. Note that $Q'A' \subset A'$. There is a Q' invariant, ergodic measure on $[0,1]$ which is equivalent to Lebesgue measure [**12**] (the explicit form of this measure is given in [5,10]). Thus the Lebesgue measure of A' is either 0 or 1. Hence the Lebesgue measure of A is either 0 or $\beta^{-1} - 1$.

Suppose the second is true, so that $h(y) = 0$ almost everywhere on $[2 - \beta^{-1}, 1] \supset [\beta, 1]$. By symmetry $h(y) = 0$ almost everywhere on $[0, \beta^{-1} - 1] \supset [0, 1 - \beta]$. For $y < 1 - \beta$, $h(y) = 0$ implies $h(y/\beta) = 0$ ((3.1), (3.3)). Accordingly $h(y) = 0$ almost everywhere on $[0, 1 - \beta]$ implies by induction that $h(y) = 0$ almost everywhere on $[0, \beta^{-i-1} - \beta^{-i}]$ and thus almost everywhere on $[0, 1]$. This is an obvious contradiction. That is, the Lebesgue measure of A is zero and $h(y) > 0$ almost everywhere in $[0, \beta^{-1} - 1] \cup [2 - \beta^{-1}, 1]$. Now let $[0, \alpha]$ be the largest interval such that $h(y) > 0$ almost everywhere in $[0, \alpha]$. Either $\alpha = 1$ or $1 - \beta < \alpha < \beta$. In the second case, $h(y) = 0$ on a set of positive measure in $[\alpha, \alpha + \epsilon] \subset [\alpha, \beta]$, and then (3.3) implies $h(y) = 0$ on a set of positive measure in $[0, \alpha]$. Thus $\alpha = 1$ and $h(y) > 0$ almost everywhere. This proves the theorem.

4. Jacobians

If T is a map from a measure space with measure μ to one with measure ν, the Jacobian T' is the derivative $d\nu T/d\mu$. This definition is valid for transformations T for which: (i) T^{-1} is countable for each x, (ii) $T(A)$ is measurable if A is measurable (T is positively measurable), and (iii) $T(A)$ has measure zero if A has measure zero (T is positively non-singular). A general almost everywhere countable-to-one T can be replaced by a conjugate T which satisfies (i)–(iii). However to compute with an explicit T, one needs to know (i)–(iii) hold. In the present case, Theorem 3 has the following corollary.

Lemma. *As defined in* (1.1), T_β *satisfies* (i)–(iii) *if* σ_β *is absolutely continuous with respect to Lebesgue measure.*

Proof of Theorem 2. Consider the sets

$$L_1 = \{(x, y): 0 \le x < \tfrac{1}{2}, 0 \le y \le (1 - \beta)/\beta\},$$
$$R_1 = \{(x, y): \tfrac{1}{2} \le x \le 1, (2\beta - 1)/\beta \le y \le 1\}.$$

Note that R_1 and L_1 are symmetric images of each other through the center of the square, so that $\mu_\beta(l_1) = \mu_\beta(R_1)$. Clearly T_β is one-to-one on these sets and so $T'_\beta = 1$ on these sets. We claim that up to a set of μ_β-measure 0,

$$L_1 \cup R_1 = \{T'_\beta = 1\} \qquad (4.1).$$

If not, we may suppose there is a set A of the form $A_v \times [0, \frac{1}{2}]$ with $A_v \subset [1 - \beta, 1]$ such that $\mu_\beta A = \frac{1}{2}\sigma_\beta A_v \neq 0$ and $T'_\beta = 1$ on A, so that $\mu_\beta(T_\beta A) = \mu_\beta(A)$. Then, since T_β is measure-preserving, $\mu_\beta(T_\beta^{-1} T_\beta A \cap ([\frac{1}{2}, 1] \times [0, 1])) = 0$. This set has the form $\tilde{A}_v \times ([\frac{1}{2}, 1])$, where $\tilde{A}_v = A_v - (1 - \beta)/\beta^{-1}$. By Theorem 3, if $\sigma_\beta A_v \neq 0$, then $\sigma_\beta \tilde{A}_v \neq 0$. This proves (4.1).

Let $j(\beta) = \mu_\beta(L_1)$. We next claim that $j(\beta)$ *is a strictly decreasing function of* β, $\frac{1}{2} \le \beta < 1$. From (2.1),

$$\begin{aligned}\sigma_\beta^{(N)}[\beta, 1] &= 2^{-(N+1)}[\#\{(a_0, a_1, \ldots, a_N): \sum_{i=0}^{N} a_i\beta^i \le \beta^{-1}, \text{ each } a_i = 0 \text{ or } 1, i = 0, \ldots, N\}] \\ &= 1 - 2^{-(N+1)}[\#\{(a_0, a_1, \ldots, a_N): \sum_{i=0}^{N} a_i\beta^i > \beta^{-1}, \text{ each } a_i = 0 \text{ or } 1, i = 0, \ldots, N\}] \\ &= 1 - 2^{-(N+1)}[\#\{(a_0, a_1, \ldots, a_N): \sum_{i=0}^{N} a_i\beta^{i+1} > 1, \text{ each } a_i = 0 \text{ or } 1, i = 0, \ldots, N\}].\end{aligned}$$

Since $\sum_{i=0}^{N} a_i\beta^{i+1}$ is increasing as a function of β, the number of terms contributing to the last expression increases as β increases, so that $j(\beta)$ decreases.

To prove that $j(\beta)$ decreases *strictly*, we construct a sequence $\{a_i\}$, $i = 1, 2, \ldots, \infty$, such that $\sum_{i=0}^{\infty} a_i\beta^{i+1} = 1$. Inductively let $a_i = 0$ if $\sum_{j=0}^{i-1} a_j\beta^{j+1} \geq 1$ and let $a_i = 1$ otherwise. We claim that for each i,

$$\sum_{j=0}^{i} a_j\beta^{j+1} \geq 1 - \beta^{i+1}. \tag{4.2}$$

Equation (4.2) is true for $i = 0$, since $1 - \beta \leq \beta$. Use induction on i. By the inductive definition, (4.2) is certainly true if $a_i = 0$. If $a_i = 1$, then by the inductive assumption,

$$\sum_{j=0}^{i} a_j\beta^{j+1} \geq 1 - \beta^i + \beta^{i+1} \geq 1 - \beta^i + \beta^{i+1} = 1 - \beta^i(1-\beta) \geq 1 - \beta^{i+1}$$

since $1 - \beta \leq \beta$. Thus (4.2) is proved, which in turn shows that for this particular sequence of a_i, $\sum_{i=0}^{\infty} a_i\beta^{i+1} = 1$.

For any $\tilde{\beta} > \beta$, $\sum_{i=0}^{\infty} a_i\tilde{\beta}^{i+1} > 1$. Therefore there exists N such that $\sum_{i=0}^{N} a_i\tilde{\beta}^{i+1} > 1$. Hence $\sigma_{\beta}^{(M)}[\tilde{\beta}, 1] \leq \sigma_{\beta}^{(M)}[\beta, 1] - 2^{-N}$ for $M \geq N$ and $j(\beta)$ is strictly decreasing. Since $\mu_\beta(\{T'_\beta = 1\}) = 2j(\beta)$ if μ_β is absolutely continuous, $\mu_\beta(\{T'_\beta = 1\})$ is strictly decreasing in these cases.

We compute

$$\begin{aligned}
&\int_{[0,1]\times[0,1]} \frac{d\mu_\beta T_\beta}{d\mu_\beta}\, d\mu_\beta \\
&= \int_{[0,\frac{1}{2}]\times[0,1]} \frac{d\mu_\beta T_\beta}{d\mu_\beta}\, d\mu_\beta + \int_{[\frac{1}{2},1]\times[0,1]} \frac{d\mu_\beta T_\beta}{d\mu_\beta}\, d\mu_\beta \\
&= \int_{[0,1]\times[0,1-\beta]} d\mu_\beta + \int_{[0,1]\times[\beta,1]} d\mu_\beta \\
&= 2(1 - j(\beta)).
\end{aligned}$$

The penultimate equality holds because T_β is positively nonsingular. Thus this integral is a strictly increasing function of β for μ_β absolutely continuous.

Finally we compute the integral of the logarithm. Note that $\log T'_\beta > 0$ almost everywhere. Recall that $h(y)$ is the Radon-Nikodym derivative of μ_β with respect to Lebesgue measure $dx\,dy$. Note that since $h(y) > 0$ almost everywhere

$$\frac{d\mu_\beta T_\beta}{d\mu_\beta} = \frac{d\mu_\beta T_\beta}{dx\,dyT_\beta}\,\frac{dx\,dyT_\beta}{dx\,dy}\,\frac{dx\,dy}{d\mu_\beta} = 2\beta\frac{h(T_\beta\,\cdot)}{h(\cdot)}.$$

Moreover, with respect to Lebesgue measure $dx\,dy$, the Jacobian of T_β equals the 'actual' Jacobian 2β. Thus by the ergodic theorem,

$$\int_{[0,1]\times[0,1]} \log T'_\beta\, d\mu_\beta = \lim_{n\to\infty}\left[\log(2\beta) + \left(\frac{\log h(T_\beta^n(x,y)) - \log h(x,y)}{n}\right)\right] = \log(2\beta)$$

for almost all (x, y). This completes the proof of Theorem 2.

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Weakly Mixing Actions of F^{∞} have infinite subgroup actions which are Bernoulli.

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Abstract

Let F be a finite field, and F^{∞} the direct sum of countably many copies of F. Regarding F^{∞} as a vector space over F, we extend the multiple recurrence theory of weakly mixing $\mathbb{Z}$-actions to weakly mixing actions of F^{∞}. From this we argue that such a weakly mixing action must have a subgroup action, isomorphic to F^{∞}, that is Bernoulli.

Introduction:

The central results of this paper were motivated by the work of U. Krengel "Weakly Wandering Vectors and Weakly Wandering Partitions" [K].

Given an isometry U of a complex Hilbert space $\mathcal{H}$, a vector $f \in \mathcal{H}$ is called weakly wandering if there exists a sequence $0 = k_0 < k_1 < k_2 \ldots$ with $U^{k_i}(f)$, $i \in \mathbb{N}$ pairwise orthogonal. Krengel showed that such an isometry has continuous spectrum if and only if weakly wandering vectors are dense.

This result translates into ergodic theory as follows. Let T be a measure-preserving transformation of a probability space $(X,\mathcal{F},\mu)$. Consider the sets $A \in \mathcal{F}$ for which there is a sequence $0=k_0<k_1<k_2\ldots$ with the sets

$$T^{-n_i}(A),\ i \in \mathbb{N},\ \text{pairwise independent.}$$

T is weakly mixing if and only if such sets are dense in $\mathcal{F}$.

This reformulation of Krengel's result is open to a natural strengthening. Let P be a finite, measurable partition of X. P is called weakly independent (with respect to T) if there exists a sequence $0=k_0<k_1<k_2$ so that the partitions

$$T^{-k_0}(P),\ T^{-k_1}(P)\ ,\cdots,\ \text{are i.i.d.}$$

Krengel defined 2-sided weakly mixing of T as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left|\mu(T^{-k}(A)\cap B\cap T^{k}(C)) - \mu(A)\mu(B)\mu(C)\right| = 0.$$

He showed that weakly mixing is necessary for density of the weakly independent partititions and 2-sided weakly mixing is sufficient.

Note: the distance between $P = \{A,\cdots,A_n\}$ and $\bar{P} = \{\bar{A},\cdots,\bar{A}_n\}$ is $\sum_{i=1}^{n} \mu(A_i\Delta\bar{A}_i)$.

Krengel also conjectured that weakly mixing was both necessary and sufficient.

Furstenberg proved this by showing weakly mixing was weakly mixing of all orders [F].

The present paper is devoted to the study of analogous results for measure preserving actions of the infinite direct sum of copies of a finite field F. We denote this group F^{∞}.

The study of measure preserving actions of such groups has led recently to interesting combinational applications. In particular a multiple recurrence theorem analogous to Furstenberg's ergodic Szemeredi theorem is true, and allows one to prove a density version of the so-called geometric Ramsey theorem (see [B] section 3).

What we show here is that if $\{T_g\}_{g\in F^{\infty}}$ is a weakly mixing action then not only are weakly mixing partitions dense, but the independent takes place along an infinite subgroup of F^{∞}, isomorphic to F^{∞} Moreover this subgroup action is in fact Bernoulli.