Jerome Malitz

Introduction to Mathematical Logic

Set Theory
Computable Functions
Model Theory

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Model Theory

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Preface

This book is intended as an undergraduate senior level or beginning graduate level text for mathematical logic. There are virtually no prerequisites, although a familiarity with notions encountered in a beginning course in abstract algebra such as groups, rings, and fields will be useful in providing some motivation for the topics in Part III.

An attempt has been made to develop the beginning of each part slowly and then to gradually quicken the pace and the complexity of the material. Each part ends with a brief introduction to selected topics of current interest.

The text is divided into three parts: one dealing with set theory, another with computable function theory, and the last with model theory. Part III relies heavily on the notation, concepts and results discussed in Part I and to some extent on Part II. Parts I and II are independent of each other, and each provides enough material for a one semester course.

The exercises cover a wide range of difficulty with an emphasis on more routine problems in the earlier sections of each part in order to familiarize the reader with the new notions and methods. The more difficult exercises are accompanied by hints. In some cases significant theorems are developed step by step with hints in the problems. Such theorems are not used later in the sequence.

The part dealing with set theory is intended to provide a notational and conceptual framework for areas of mathematics outside of logic as well as to introduce the student to those topics that are of particular interest to those working in the foundations of set theory.

We hope that the part of the text devoted to computable functions will be of interest to those who intend to work with real world computers.

We believe that the notation, methodology, and results of elementary logic should be a part of a general mathematics program and are of value in a wide variety of disciplines within mathematics and outside of mathematics.

Boulder, Colorado
March 1979

J. MALITZ

Glossary of Symbols

$\{\cdots\}$	2
$\mathbf{I}$	2
$\mathbf{N}$	2
$\mathbf{N}^+$	2
$\mathbf{Q}$	2
$\mathbf{Q}^+$	2
$\mathbf{R}$	2
$\mathbf{R}^+$	2
$\{x: \cdots\}$	2
$\varnothing$	2
$\in$	2
$\subseteq, \supseteq, \subset, \supset$	2, 3
$\cup$	3
$\bigcup X$	4
$\bigcup_{i \in I} A_i$	4
$\cap$	4
$\bigcap X$	4
$B - A$	4
$P(X)$	5
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$f: A \to B$	7
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$P(\bar{n}, x)$	75
Prime	75

Prim	75
Exp′	76
Max	77
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PART I

An Introduction to Set Theory

1.1 Introduction

Through the centuries mathematicians and philosophers have wondered if size comparisons between infinite collections of objects can be made in a meaningful way. Does it make sense to ask if there are as many even numbers as odd numbers? What does it mean to say that one infinite collection has greater magnitude than another? Can one speak of different sizes of infinity?

Before the last three decades of the nineteenth century, mathematicians and philosophers generally agreed that such notions are not meaningful. But then in the early 1870s, a German mathematician, Georg Cantor (1845–1918), in a remarkable series of papers, formulated a theory in which size comparisons between infinite collections could be made. This theory became known as set theory. As with many radical departures from traditional approaches, his ideas were at first violently attacked but now have come to be regarded as a useful and basic part of modern mathematics. This chapter is an introduction to set theory.

1.2 Sets

We use the term *set* to refer to any collection of objects. The objects composing a set will be referred to as the *members* or *elements* of the set. There are various ways to denote sets. One approach is to list the elements of the set in some way and enclose this list in braces. For example, using this convention, the set consisting of the numbers 1, 2, and 3 is denoted by $\{1,2,3\}$. A set is completely determined by its members, and so the order in which we list the elements is immaterial. Thus $\{1,2,3\}=\{2,3,1\}=\{3,2,1\}=\{1,3,2\}=\{2,1,3\}=\{3,1,2\}$.

A set may have so many members belonging to it that it is impractical or impossible to use the above method of notation, and so other notational devices must be used. For example, instead of using the method described above to denote the set of all positive integers less than or equal to 10^{10}, we might use $\{1,2,3,\ldots,10^{10}\}$ to denote this set. The three dots indicate that some members of the set being described have not been listed explicitly. Of course, in using this notational device it is important to include enough members of the list before and after the three dots so that the reader will know which elements belong to the set and which do not. For example, the set of even integers between -100 and 100 inclusive should not be denoted by $\{-100,\ldots,100\}$ but by something like $\{-100,-98,\ldots,-4,-2,0,2,4,\ldots,98,100\}$ or by $\{0,2,-2,4,-4,\ldots,98,-98,100,-100\}$. Again, the order in which the elements are listed is arbitrary as long as the reader understands which elements of the set have not been mentioned explicitly.

The method for denoting sets using the three dots abbreviation can also be used for infinite sets. For example, the set of even integers can be denoted by $\{0,2,-2,4,-4,\ldots\}$ or by $\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$. We will use $\mathbf{N}$ to denote the set $\{0,1,2,3,\ldots\}$ of natural numbers, while $\mathbf{N}^+$ will denote $\{1,2,3,\ldots\}$. $\mathbf{I}$ will denote the set of integers $\{0,1,-1,2,-2,3,-3,\ldots\}$. $\mathbf{Q}$ will denote the set of rationals, $\mathbf{Q}^+$ the set of positive rationals, $\mathbf{R}$ the set of real numbers, and $\mathbf{R}^+$ the set of positive reals.

If a set consists of exactly those objects satisfying a certain condition, say P, we may denote it by $\{x:P(x)\}$, which is read: "the set of all x such that $P(x)$ is true." For example, $\{x:3\leqslant x\leqslant 8$ and x is a rational number$\}$ is the set of rationals between 3 and 8 inclusive. Notice that x merely represents a typical object in the set under consideration, and any letter will serve just as well in place of x. Thus $\{1,2,3\}=\{x:x$ is an integer and $1\leqslant x\leqslant 3\}=\{y:y$ is an integer and $1\leqslant y\leqslant 3\}=\{x:x$ is an integer and $0<x<4\}$. Notice that the last two conditions are different but define the same set.

We consider as a set the collection which has no members. We call this set the null set and denote it by $\varnothing$, rather than $\{\ \}$.

A set may contain other sets as elements. For example, the set $\{1,\{2,3\}\}$ is the set whose elements are the number 1 and the set $\{2,3\}$. It is important to understand that this set has only two elements, namely 1 and $\{2,3\}$. 2 is an element of $\{2,3\}$, but 2 is not an element of $\{1,\{2,3\}\}$.

We write $x\in A$ when x is an element of A, and $x\notin A$ otherwise.

Let A be a set. We say that a set B is a *subset* of A if each element of B is an element of A. If B is a subset of A we write $B\subseteq A$ or $A\supseteq B$. If $B\subseteq A$

and, in addition, $A \neq B$, we write $B \subset A$ or $A \supset B$ and say that B is a proper subset of A. So $\{1,\{2,3\}\} \subset \{1,\{2,3\},4\}$ but $\{1,\{2,3\}\} \not\subseteq \{1,\{2\},\{3\}\}$.

Notice that $A \subseteq A$ and $\varnothing \subseteq A$ for every set A (since $\varnothing$ has no elements, it is true that every element of $\varnothing$ is an element of A). Another trivial observation is that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

We note that if A and B are sets such that $A \subseteq B$ and $B \subseteq A$, then $A = B$. For if $x \in A$, then since $A \subseteq B$, $x \in B$. Similarly, if $y \in B$ we have $y \in A$. Thus A and B contain precisely the same elements and so are equal. This will be used frequently in what follows; two sets A and B will be shown to be equal by proving both $A \subseteq B$ and $B \subseteq A$.

Next we consider ways of combining sets to get new sets.

The *union* of A and B, denoted by $A \cup B$, is the set whose elements belong either to A or to B. In other words $A \cup B = \{x : x \in A \text{ or } x \in B\}$. (In mathematics we use the word "or" in the inclusive sense. So when we say that an object is in A or in B we include the case where the object is in both A and B.) For example

$$\{1,2\} \cup \{3,4\} = \{1,2,3,4\},$$
$$\{a,b,c\} \cup \{a,c,d\} = \{a,b,c,d\},$$
$$\{x : x \text{ is an even integer}\} \cup \{x : x \text{ is an odd integer}\}$$
$$= \{x : x \text{ is an integer}\}.$$

Some of the elementary properties of the union operation are summarized below.

Theorem 2.1.

i. *$A \subseteq B$ implies that $A \cup B = B$.*
ii. *$A \cup B = B \cup A$.*
iii. *$A \cup (B \cup C) = (A \cup B) \cup C$.*

The proof of the theorem is very easy, and we leave all parts but iii as exercises.

To prove part iii, first suppose that $x \in A \cup (B \cup C)$. Then either $x \in A$ or $x \in B \cup C$. If $x \in A$, then $x \in A \cup B$, and so $x \in (A \cup B) \cup C$. If $x \in B \cup C$, then $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cup B$, and so $x \in (A \cup B) \cup C$. If $x \in C$, then $x \in (A \cup B) \cup C$. Hence we have shown that whenever $x \in A \cup (B \cup C)$, then $x \in (A \cup B) \cup C$, in other words, we have shown that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. In the same way one proves that $A \cup (B \cup C) \supseteq (A \cup B) \cup C$ (the reader should check this). Hence $A \cup (B \cup C) = (A \cup B) \cup C$ as claimed.

Because of part iii, no confusion can arise if parentheses are omitted from $(A \cup B) \cup C$ and we write $A \cup B \cup C$.

It should be clear what is meant by $A_1 \cup A_2 \cup \dots \cup A_n$, namely, $\{x : x \in A_1 \text{ or } x \in A_2 \text{ or } \dots \text{ or } x \in A_n\}$. An alternative notation for this set is

$\bigcup \{A_i : i \in \mathbf{N}^+ \text{ and } i \leqslant n\}$. In general, if X is a non-empty set of sets, then $\bigcup X = \{y : \text{there is a } Y \in X \text{ such that } y \in Y\}$. This is called the *union over* X. So if $X = \{A_1, A_2, \ldots, A_n\}$, then $\bigcup X = A_1 \cup A_2 \cup \ldots \cup A_n$. For example, if $A_i = \{x : x = i/n \text{ for some } n \in \mathbf{N}^+\}$ (so that $A_5 = \{5/1, 5/2, 5/3, 5/4, \ldots\}$), then $\bigcup \{A_i : i \in \mathbf{N}^+\} = \mathbf{Q}^+$. Instead of writing $\bigcup \{A_i : i \in I\}$ we may write $\bigcup\limits_{i \in I} A_i$.

The *intersection* of A and B, $A \cap B$, is the set whose elements are simultaneously elements of A and of B. In other words $A \cap B = \{x : x \in A \text{ and } x \in B\}$. For example $\{1,3,9\} \cap \{1,5,9\} = \{1,9\}$ and $\{x : x \in \mathbf{R}^+ \text{ and } x < 5\} \cap \{x : x \in \mathbf{Q} \text{ and } x \geqslant 3\} = \{x : x \in \mathbf{Q} \text{ and } 3 \leqslant x < 5\}$.

Theorem 2.1. (Cont.)

i′. *$A \subseteq B$ implies that $A \cap B = A$.*
ii′. *$A \cap B = B \cap A$.*
iii′. *$A \cap (B \cap C) = (A \cap B) \cap C$.*

The proofs are very easy and left as exercises.

As in the case of the union operation, the intersection operation generalizes to the intersection over a set of sets. Letting X be a non-empty set of sets, we define the *intersection* over X, $\bigcap X$, to be $\{y : \text{for all } Y \in X, y \in Y\}$. So if $X = \{A_1, A_2, \ldots, A_n\}$, then $\bigcap X = A_1 \cap \ldots \cap A_n$. (As before, we use iii′ to justify our omission of parentheses in $A_1 \cap \ldots \cap A_n$.) As another example let $A_n = \{x : x \in \mathbf{R} \text{ and } |x| < 1/n\}$. Let $X = \{A_n : n \in \mathbf{N}^+\}$. Then $\bigcap X = \{0\}$.

We say that A and B are *disjoint* if $A \cap B = \varnothing$. Similarly, X is a set of *pairwise disjoint sets* if for all $A, B \in X$, either $A = B$ or $A \cap B = \varnothing$.

We next state some easily proved facts relating union and intersection. The proofs are left for the exercises.

Theorem 2.1. (Cont.)

iv. *$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, and more generally $\left(\bigcup X\right) \cap \left(\bigcup Y\right) = \bigcup \, (A \cap B : A \in X \text{ and } B \in Y)$.*
iv′. *$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, and more generally $\left(\bigcap X\right) \cup \left(\bigcap Y\right) = \bigcap \, (\{A \cup B : A \in X \text{ and } B \in Y\})$.*

The *difference of A from B*, denoted $B - A$, is the set of elements in B but not in A; in other words we define $B - A = \{x : x \in B \text{ and } x \notin A\}$. For example, $\mathbf{Q}^+ - \{x : x \in \mathbf{R} \text{ and } x \leqslant 3\}$ is the set of positive rationals greater than 3. As another example, $\{1,4,9\} - \{3,4,8\} = \{1,9\}$. $B - A$ is also called the *complement of A in B*.

We next state several relations between the above notions.

Theorem 2.1. (Cont.)

v. $A \subseteq B$ *implies* $B-(B-A)=A$.
vi. $C \supseteq B \supseteq A$ *implies* $C-A \supseteq C-B$.
vii. $C-(A \cup B)=(C-A) \cap (C-B)$, *and more generally* $C-\left(\bigcup X\right)=\bigcap \{C-A : A \in X\}$.
viii. $C-(A \cap B)=(C-A) \cup (C-B)$, *and more generally* $C-\left(\bigcap X\right)=\bigcup \{C-A : A \in X\}$.

We prove vii, leaving the proof of the other clauses for the exercises. Here and throughout the text we use 'iff' to abbreviate 'if and only if'.

$$x \in C-\left(\bigcup X\right) \quad \text{iff}$$

$$x \in C \text{ and } x \notin A \quad \text{for all } A \in X \quad \text{iff}$$

$$x \in C-A \quad \text{for all } A \in X \quad \text{iff}$$

$$x \in \bigcap \{C-A : A \in X\}.$$

In other words, $C-\left(\bigcup X\right)$ and $\bigcap \{C-A : A \in X\}$ have the same members and so are identical, as claimed in vii.

Clauses vii and viii are called De Morgan's rules.

We next define the *power set*, $P(X)$, of a set X. This is the set of all subsets of X, i.e., $P(X)$ is defined as $\{Y : Y \subseteq X\}$.

For example, if $X=\{1,2,3\}$, then $P(X)=\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Clearly, we always have $\phi \in P(X)$ and $X \in P(X)$. Elementary properties of the power set operation will be found in Exercise 7 below.

EXERCISES FOR §1.2

1. How many elements are there in each of the following sets?

$$\{1,2,\phi\}, \quad \{1,\{1,\phi\}\}, \quad \{\phi\}, \quad \{1\}, \quad \{\{1\}\}.$$

2. Which of the following are true?

$$\varnothing \in \varnothing, \qquad \varnothing \subseteq \varnothing, \qquad \{1\} \in \{1,2\}, \qquad 1 \in \{\{1\},2\}.$$

3. Show that
   (a) if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, and
   (b) if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

4. Supply the missing proofs for Theorem 2.1.

5. List the elements of $P(\{1,2,3,4\})$.

6. List the elements of $P(P(P(\varnothing)))$.

7. Show that
   (a) $A \supseteq B$ implies $P(A) \supseteq P(B)$;
   (b) $P(A \cup B) \supseteq P(A) \cup P(B)$, and more generally

$$P\left(\bigcup X\right) \supseteq \bigcup \{P(A) : A \in X\};$$

   (c) $P(A \cap B) \subseteq P(A) \cap P(B)$, and more generally

$$P\left(\bigcap X\right) \subseteq \bigcap \{P(A) : A \in X\};$$

When does equality hold in (b) and in (c)?

1.3 Relations and Functions

The aim of this section is to supply definitions of 'relation', 'function' and related notions in enough generality to be of service throughout the book. These notions ultimately rest on that of the ordered pair (a,b). Although 'ordered pair' can be defined in terms of the membership relation, as can all the notions of the classical mathematics, we will not do this until later. For the time being we shall take the ordered pair (a,b) to be an undefined notion with the property that $(a,v)=(c,d)$ if and only if $a=c$ and $b=d$. For example $(3,8) \neq (8,3)$ (although $\{3,8\}=\{8,3\}$). Similarly, the only property of n-tuples that we shall use is that $(a_1,\ldots,a_n)=(b_1,\ldots,b_n)$ iff $a_i=b_i$ for all $i \leq n$.

The *Cartesian product* of A and B, written $A \times B$, is $\{(x,y) : x \in A$ and $y \in B\}$. More generally, we define $A_0 \times A_1 \times \ldots \times A_n$ to be $\{(a_0,\ldots,a_n) : a_i \in A_i$ for each $i \in \{0,\ldots,n\}\}$. For example, $(1,\frac{3}{4}) \in \mathbf{N} \times \mathbf{Q}$, but $(\frac{3}{4},1) \notin \mathbf{N} \times \mathbf{Q}$.

If for each $i,j \in \{0,\ldots,k-1\}$ we have $B=A_i=A_j$, then we abbreviate $A_0 \times \ldots \times A_{k-1}$ by $[B]_k$. For example, $[\mathbf{R}]_n$ is Euclidean n-space.

A *binary relation* is a set of ordered pairs. For example $\{(x,y) : x<y$ and $x \in \mathbf{N}, y \in \mathbf{N}\}$ is a binary relation. So is $\{(3,4),(1,1)\}$, as well as the circle in Euclidean 2-space of radius 3 with center $(4,\pi)$, namely $\{(x,y) : (x-4)^2 + (y-\pi)^2 = 3^2\}$.

The *domain* of a binary relation R, sometimes written $\operatorname{Dom} R$, is $\{x :$ there is a y such that $(x,y) \in R\}$; the *range* of R, $\operatorname{Ran} R$, is $\{y :$ for some x, $(x,y) \in R\}$. The *field* of R is $\operatorname{Dom} R \cup \operatorname{Ran} R$. In the first of the three examples above we have $\operatorname{Dom} R = \mathbf{N}$, $\operatorname{Ran} R = \mathbf{N}^+$; in the second $\operatorname{Dom} R = \{3,1\}$, $\operatorname{Ran} R = \{4,1\}$; and in the third $\operatorname{Dom} R = \{x : 1 \leq x \leq 7\}$, $\operatorname{Ran} R = \{y : \pi - 3 \leq y \leq \pi + 3\}$. One frequently writes xRy instead of $(x,y) \in R$, and $x\not{R}y$ if $(x,y) \notin R$.

More generally, a *k-relation* is a set of ordered k-tuples (so a 2-relation is a binary relation). As an example of a 3-relation we have $\{(x,y,z) : (x,y,z) \in [\mathbf{N}]_3$ and z is the least common multiple of x and $y\}$. Another example is $\{(x,y,z) : (x,y,z) \in [\mathbf{R}]_3$ and $x+y=z\}$. We do not define the domain or range of a k-relation when $k \neq 2$.

The set of all primes is an example of a 1-relation, as is the set of all multiples of π.

A *function* f is a 2-relation such that for every x there is at most one y for which $(x,y) \in f$. In other words, if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$. When f is a function, one usually writes $f(x) = y$ instead of $(x,y) \in f$, and says that y is the value of f at x.

For example, $\{(1,3),(3,1),(\pi,1)\}$ is a function, but $\{(1,3),(3,1)\ (1,\pi)\}$ is not. $\{(x,y): x = y^3$ and $x \in \mathbf{N}$ and $y \in \mathbf{N}\}$ is a function, but $\{(x,y): x = y^2$ and $x \in \mathbf{N}$ and $y \in \mathbf{I}\}$ is not.

A function f is *one to one*, abbreviated 1-1, if $\{(y,x): f(x) = y\}$ is a function, i.e., if when $f(x) = y$ and $f(z) = y$ we have $x = z$. In our examples of functions above, the second is 1-1 but the first is not.

We say that a function f is *on* A if $\operatorname{Dom} f = A$; *into* B if $\operatorname{Ran} f \subseteq B$; *onto* B if $\operatorname{Ran} f = B$. If f is a function on A into B, we may write $f: A \rightarrow B$. The notation $f: A \overset{1\text{-}1}{\rightarrow} B$ adds the condition that f is 1-1, while $f: A \underset{\text{onto}}{\rightarrow} B$ adds the condition that f is onto B. The set of all functions on A into B is denoted by ${}^{A}B$, i.e., ${}^{A}B = \{f : f: A \rightarrow B\}$.

By $f[C]$ we mean $\{y: \text{for some } x \in C, f(x) = y\}$. Notice that no restriction is placed on C; C need not be included in $\operatorname{Dom} f$. For example, if $f = \{(x,y): y = x^2$ and $x \in \mathbf{N}\}$ and $C = \{x: x < \pi$ and $x \in \mathbf{R}\}$, then $f[C] = \{0,1,4,9\}$.

Define $f^{-1}[Y]$ to be $\{x: f(x) \in Y\}$. $f^{-1}[Y]$ is defined even if $Y \nsubseteq \operatorname{Ran} f$. So if $f(x) = 3x + 2$ for all $x \in \mathbf{R}^{+}$, then $f^{-1}[\{y: 0 < y < 11\}] = \{x: 0 < x < 3\}$. As another example, if $f(x) = x^2$ for each $x \in \mathbf{R}$, then $f^{-1}[\{y\}] = \{-\sqrt{y}, \sqrt{y}\}$ for each $y \in \mathbf{R}^{+}$. If $f: A \underset{\text{onto}}{\overset{1\text{-}1}{\rightarrow}} B$, then the set $\{(b,a): f(a) = b\}$ is a 1-1 function on B onto A which we call f *inverse*, written f^{-1}. Notice that $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = a$ for all $a \in A$ and all $b \in B$.

The *restriction of* f *to* C, abbreviated $f \upharpoonright C$, is the function g with domain $C \cap \operatorname{Dom} f$ such that for each $x \in C \cap \operatorname{Dom} f$ we have $g(x) = f(x)$. In other words $g = \{(x,y): x \in C \cap \operatorname{Dom} f$ and $y = f(x)\}$.

Notice that C is arbitrary and need not be a subset of $\operatorname{Dom} f$. For example, if $f = \{(x,y): y = x^2$ and $x \in N\}$ and $C = \{x: x < \pi$ and $x \in \mathbf{R}\}$, then $f \upharpoonright C = \{(0,0),(1,1),(2,4),(3,9)\}$.

If $g = f \upharpoonright C$ and $C \subseteq \operatorname{Dom} f$, then we say that *$f$ is an extension of g*.

Let $f \in {}^{B}C$ and let $g \in {}^{A}B$. The *composite of f and g*, written $f \circ g$, is that element of ${}^{A}C$ defined by $(f \circ g)(x) = f(g(x))$ for all $x \in A$.

Theorem 3.1. *Let $f \in {}^{B}C$, $g \in {}^{A}B$. Then*

i. *if f and g are 1-1, then so is $f \circ g$.*
ii. *if f is onto C and g is onto B, then $f \circ g$ is onto C.*

PROOF OF i. Suppose f and g are 1-1, and $(f \circ g)(a) = (f \circ g)(b)$. Then $f(g(a)) = f(g(b))$. Since f is 1-1, $g(a) = g(b)$. Since g is 1-1, $a = b$. □

We leave the proof of part ii as an exercise (see Exercise 15).

EXERCISES FOR §3.

1. What are the elements of $\{1,3\}\times\{1,\pi,4\}$?

2. If A has m elements and B has n elements, how many elements does $A\times B$ have?

3. Prove that if $A_i\subseteq B_i$ for each $i\in\{1,2,\ldots,k\}$ then $A_1\times\ldots\times A_k\subseteq B_1\times\ldots\times B_k$.

4. Show that the relation $<$ on $\mathbf{Q}$ is not a set of the form $A\times B$.

5. If the following statement is true, prove it; if not, give a counter example:
$$\text{If}\quad A\supseteq B\cup C\quad\text{then}\quad (A\times A)-(B\times C)=(A-B)\times(A-C).$$

6. Prove or disprove the following statement:
$$(A_1\times A_2)\cup(B_1\times B_2)=(A_1\cup B_1)\times(A_2\cup B_2).$$

7. Prove or disprove the following statement:
$$(A_1\times A_2)\cap(B_1\times B_2)=(A_1\cap B_1)\times(A_2\cap B_2).$$

8. For each relation R below, find $\operatorname{Dom} R$, $\operatorname{Ran} R$, and the field of R:
   (a) $R=\{(1,4),(\pi,3),(\pi,1),(1,\pi)\}$.
   (b) $R=\{(x,y):|x|+|y|=1\}$.
   (c) $R=\{p:p\in[R]_2 \text{ and } |p-(1,0)|+|p-(1,0)|=3\}$
$$\left[\text{where}\quad |(x_1,y_1)-(x_2,y_2)|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\ \right].$$

9. Which of the following are functions; which of the functions are 1-1?
   (a) $\{(x,y):x>0,\ x^2+y^2=1, \text{ and } x\in\mathbf{R},y\in\mathbf{R}\}$.
   (b) $\{(x,y):y>0,\ x^2+y^2=1, \text{ and } x\in\mathbf{R},y\in\mathbf{R}\}$.
   (c) $\{(x,y):x>0,\ y>0,\ x^2+y^2=1 \text{ and } x\in\mathbf{R},\ y\in\mathbf{R}\}$
   (d) $\{(x,y,z):x,y,z\in\mathbf{N}^+ \text{ and } z=2^x3^y\}$.
   (e) $\{(x,y,z):x,y,z\in\mathbf{N}^+ \text{ and } z=2x+3y\}$.

10. Prove:
   (a) $f\left[\bigcup X\right]=\bigcup\{f[A]:A\in X\}$.
   (b) $f\left[\bigcap X\right]\subseteq\bigcap\{f[A]:A\in X\}$.
   Show that equality need not hold in (b) by describing sets A and B and a function f such that $f[A\cap B]\neq f[A]\cap f[B]$.

11. Show that
   (a) $(f\upharpoonright C)\upharpoonright D=f\upharpoonright(C\cap D)$
   (b) $\bigcap\{f\upharpoonright C:C\in K\}=f\upharpoonright\bigcap\{C:C\in K\}$.

12. Suppose A has n elements and B has m elements. How many elements are there in ${}^AB$? Give a proof.

13. Prove:
   (a) $f^{-1}\left[\bigcup X\right]=\bigcup\{f^{-1}[A]:A\in X\}$.
   (b) $f^{-1}\left[\bigcap X\right]=\bigcap\{f^{-1}[A]:A\in X\}$.
   (c) $f^{-1}[A-B]\supseteq f^{-1}[A]-f^{-1}[B]$.
   Show that equality need not hold in (c).

14. Find functions f and g such that $\operatorname{Ran} f=\operatorname{Ran} g=\operatorname{Dom} f=\operatorname{Dom} g$ and $f\circ g\neq g\circ f$.

15. Show that if $f:B\underset{\text{onto}}{\rightarrow}C$ and $g:A\underset{\text{onto}}{\rightarrow}B$, then $f\circ g:A\underset{\text{onto}}{\rightarrow}C$.