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Preface

This volume contains a series of individual research papers in the area of applied probability and management science. Contributions are made to inventory theory, queueing and dam theory, replacement and maintenance problems, reliability structures, and capital policy. Some of the papers are parts of Ph.D. dissertations submitted to Stanford University. Several papers represent natural continuations of the studies presented in an earlier volume in this series, "Studies in the Mathematical Theory of Inventory and Production."

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THE EDITORS

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Optimal Capital Adjustment

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1. Paths of Capital Accumulation in a Stationary Environment

The possession by a firm of a stock K of capital goods yields a flow of operating profits $\pi(K)$. Against this must be offset the costs of the investment needed to accumulate the capital, and the rate of interest α which the capital used by the firm could be earning elsewhere. The *gross investment* I includes both the amount necessary to replace capital goods that have worn out (*depreciation*) and the net additions to capital stock (*net investment*). Net investment is K', where primes denote differentiation with respect to time. We assume that the depreciation of capital goods is proportional to the stock at any given time, which is equivalent to saying that any given collection of capital goods depreciates at an exponential rate. Thus, depreciation is equal to δK, where δ is a positive constant. Gross investment, as the sum of depreciation and net investment, is given by

(1) $$I = K' + \delta K .$$

We assume that both the profits of the firm from any given stock of capital and the rate of interest, i.e., the function $\pi(K)$ and the number α, are expected by the firm to remain constant over time. The firm is also supposed to have on hand at time zero a stock of capital goods K_0. Suppose that the firm chooses a capital policy $K(t)$ which prescribes the stock of capital goods at every point in the future. The policy must satisfy the initial condition

(2) $$K(0) = K_0 .$$

Further, assume that the cost of investment is simply proportional to the

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Work done with the help of the Office of Naval Research (Task 047). I am greatly indebted to my colleague Marc Nerlove for bringing this problem to my attention [6]. He was primarily interested in the determination of optimal advertising policy under dynamic conditions, but pointed out the identity of the problem with the problem of optimal capital policy.

magnitude, i.e., that the price of investment goods does not vary with the rate of investment. The price of investment goods is assumed also to be constant in time. Without loss of generality, we may assume, then, that the rate of expenditure on capital goods is $I(t)$, which is determined from the policy $K(t)$ by (1). The net surplus of receipts over expenditures at any time t is $\pi[K(t)] - I(t)$. The present value of the policy $K(t)$ is

$$V\{K(t)\} = \int_0^{+\infty} e^{-\alpha t}\{\pi[K(t)] - I(t)\}\, dt\,. \tag{3}$$

Here V is a functional depending on the whole policy $K(t)$; the braces are used to remind us of this fact. A firm will, of course, seek to maximize V by a suitable choice of $K(t)$.

It is important to observe that we cannot restrict ourselves to differentiable or even continuous solutions for $K(t)$. Consider, for example, the simplest case, where $K(t)$ can be chosen without any restriction. Since in this case the initial holding does not constitute an effective constraint, the optimal choice of $K(t)$ at any time t is made under the same conditions; therefore it is a constant. In general, of course, this constant need not equal K_0, and there will be a discontinuity at $t = 0$. This means an instantaneous purchase of capital goods (or a sale, if K_0 is greater than the optimal stationary solution), which implies an infinite rate of investment (or disinvestment) at $t = 0$. For such paths, the integral in (3) must be interpreted with some care. In particular, $I(t)$ becomes infinite at discontinuities, and the integral can be evaluated as the limit of integrals in which $I(t)$ has increasingly large values.

The structure of the solution will depend on the nature of the function $\pi(K)$ and the restrictions on the rate of gross investment. Under perfectly competitive conditions, the latter should be unrestricted, but in practice we know that the sale of capital goods cannot be made under the same conditions as their purchase. Usually, the second-hand price is much below the price of an equivalent magnitude of new capital goods. For simplicity, we shall make the extreme assumption that the sale of capital goods is impossible, so that gross investment must be nonnegative:

$$I(t) \geqq 0\,. \tag{4}$$

This assumption will also be valid for a monopolist using specialized capital goods, since then there are no alternative buyers, or for national planning in a closed economy, where $\pi(K)$ would represent national income attributable to a given stock of capital.

For the main results, this is the only restriction that will be considered. But it may also be reasonable to assume that there is an upper bound on the rate of gross investment. This condition is particularly appropriate to the case of a monopolist or of a nation, for then the rate of gross investment is restricted by the capacity of the capital-goods industry. A more general assumption is that the cost of investment is a nonlinear function of the rate. An upper bound on the rate of investment can be replaced by an increase in cost approaching infinity, a lower bound by the assumption that cost ceases to decrease or decreases very slowly as gross investment drops below zero. (For negative values of gross investment, "cost" means the negative of the sale price of capital goods). The most general case is not discussed here; the solution when there is an upper bound on gross investment is sketched in section 5.

The assumptions on $\pi(K)$ will be stated directly in terms of conditions on a closely related function, the *net profits*:

$$P(K) = \pi(K) - (\alpha + \delta)K\,. \tag{5}$$

The relevance of this function will become clear in the following section, but its economic meaning is clear and corresponds to ordinary accounting practice; from the operating profits must be subtracted the interest on the capital and the depreciation of the capital goods to arrive at a true figure for profits. In this paper, only very minimal regularity conditions are imposed on $P(K)$. Specifically, we make two assumptions:

$$P(K) \text{ is decreasing for sufficiently large } K\,, \tag{6}$$

and

$$P(K) \text{ has a finite number of local maxima}\,. \tag{7}$$

For simplicity of exposition, we add one more:

$$\text{If } K_1 \text{ and } K_2 \text{ are both local maxima of } P(K), \text{ then } P(K_1) \neq P(K_2)\,. \tag{8}$$

The removal of (8) would lead only to inessential complications.

In many economic situations, it is reasonable to suppose that $\pi(K)$, and therefore $P(K)$, is concave (diminishing returns to scale); in that case, $P(K)$ would have only one local maximum, which would be the global maximum. Even with an initial phase of increasing returns to scale, the function $P(K)$ might have a unique maximum. In this case, as we shall see in the next section, the solution has an extremely simple, almost trivial, form.

However, if the production process has several aspects that show economies of scale in different ways, we might have a wavelike form for $P(K)$. The economies of scale might be exhausted in one area of the firm's activities before they have begun to become significant in another. In the case of a monopolist, there may be two profit maxima: one, caused by the inelasticity of demand, at a level of output and capital so low that there are no significant economies of scale; a second and higher one at a level where economies of scale become sufficient to compensate for a lower demand price.

The situation studied here is similar to that discussed in chapter 7 of [2], but the assumptions differ considerably. There, demand was given but varied in time, and depreciation was assumed to be zero. The methods used here are closely related to those of chapters 4–7 of [2].

There is another, rather different, economic interpretation of the model studied here. Suppose that the effect of advertising on shifts in demand is cumulative. We may say that the demand curve and therefore the profits at any time depend on a stock of accumulated *good will*, G. In the absence of advertising, the good will tends to disappear gradually, but it can be increased by advertising, $a(t)$. A simple hypothesis of this type would be

$$G' = a - \delta G ; \tag{9}$$

that is, good will is added to at a rate proportional to the amount of advertising, and decays exponentially. Profits at any time are a function $\pi(G)$, so that the net surplus of receipts over expenditures at any time is $\pi(G) - a$, and the present value associated with a planned advertising policy is

$$\int_0^{+\infty} e^{-\alpha t}\{\pi[G(t)] - a(t)\}\, dt .$$

Since advertising is necessarily nonnegative, the problem of optimal advertising policy is abstractly identical with that of optimal capital policy.

2. Preliminary Observations

The problem is that of maximizing $V\{K(t)\}$, defined in (3), subject to the constraints (1), (2), and (4), where the function $\pi(K)$ satisfies conditions (5)–(8). Since K_0 is a datum, the problem is equivalent to maximizing the *surplus* $W\{K(t)\}$, where

$$W\{K(t)\} = V\{K(t)\} - K_0 . \tag{10}$$

This latter form will turn out to be more convenient.

If we substitute from (1) in (3) and then in (10), we have

$$W\{K(t)\} = \int_0^{+\infty} e^{-\alpha t}\{\pi[K(t)] - \delta K(t)\}\, dt - K_0 - \int_0^{+\infty} e^{-\alpha t} K'(t)\, dt .$$

The last term will be evaluated by integration by parts:

$$\begin{aligned}\int_0^{+\infty} e^{-\alpha t} K'(t)\, dt &= [e^{-\alpha t} K(t)]_0^{+\infty} + \alpha \int_0^{+\infty} e^{-\alpha t} K(t)\, dt \\ &= \lim_{t \to +\infty} [e^{-\alpha t} K(t)] - K(0) + \alpha \int_0^{+\infty} e^{-\alpha t} K(t)\, dt .\end{aligned}$$

It should be remarked that this integration remains valid even when $K(t)$ has discontinuities; each jump in $K(t)$ contributes a term

$$e^{-\alpha t}[K(t+0) - K(t-0)]$$

to the left-hand integral, but these terms also occur in the right-hand expression. We also assume, for the moment, that the limit in the first term on the right-hand side exists. Then, recalling (2) and (5), we have

$$W\{K(t)\} = \int_0^{+\infty} e^{-\alpha t} P[K(t)]\, dt - \lim_{t \to +\infty} [e^{-\alpha t} K(t)] . \tag{11}$$

Now we observe that under the assumptions made, for any unbounded policy $K(t)$ we can find another such policy that is bounded and has a higher value for $W\{K(t)\}$. For, from (6), we can choose $\tilde{K}$ so that $\tilde{K} \geqq K_0$ and $P(K)$ is decreasing for $K \geqq \tilde{K}$. Then define

$$\tilde{K}(t) = \min [\tilde{K}, K(t)] .$$

It is easy to verify that conditions (2) and (4) are satisfied by $\tilde{K}(t)$. By construction $P[\tilde{K}(t)] \geqq P[K(t)]$ for all t, with the strict inequality holding for some interval; and, since $\tilde{K}(t)$ is bounded,

$$\lim_{t\to+\infty} e^{-\alpha t}\tilde{K}(t) = 0 \leqq \lim_{t\to+\infty} e^{-\alpha t}K(t) ,$$

so that $W\{\tilde{K}(t)\} > W\{K(t)\}$. If

$$\lim_{t\to+\infty} [e^{-\alpha t}K(t)]$$

does not exist, $W\{K(t)\}$ is, strictly speaking, not defined. But it can be said that the present value associated with $K(t)$ does not in any case exceed

$$\int_0^{+\infty} e^{-\alpha t}P[K(t)]\, dt - \liminf_{t\to+\infty} [e^{-\alpha t}K(t)] ,$$

and by the same argument we can do better with a bounded policy. Hence we can assume that

(12) $$K(t) \text{ is bounded} ,$$

and we can write

(13) $$W\{K(t)\} = \int_0^{+\infty} e^{-\alpha t}P[K(t)]\, dt .$$

It is obvious from (13) that in choosing an optimal policy the aim is to make $P[K(t)]$ as large as possible at each time t, the only constraints being those implied by (1), (2), and (4). In the absence of (4), (1) would not be an effective constraint; and since the value of the integral in (13) is clearly unaltered by changing the value of $K(t)$ at one point, $t = 0$, the constraint (2) is irrelevant. Hence the optimal policy is simply to make $P[K(t)]$ as large as possible. Now, in view of (6), $P(K)$ has a global maximum $\bar{K}$, and in view of (8), it is unique. Then the optimal policy is the *stationary* policy defined by

(14) $$K(t) \equiv \bar{K} \qquad (t > 0) .$$

The surplus associated with this policy is

(15) $$W\{\bar{K}\} = P(\bar{K})\int_0^{+\infty} e^{-\alpha t}\, dt = \frac{P(\bar{K})}{\alpha} ,$$

and this same expression is valid for any stationary policy.

This policy does satisfy all constraints for $t > 0$, since $K' \equiv 0$ and $I \equiv \delta\bar{K} > 0$. The only place it might fail to satisfy them is at $t = 0$. If $K_0 > \bar{K}$, then $K(t)$ has a downward jump at $t = 0$, which implies that $K'(0) = -\infty$, and therefore

$$I(0) = K'(0) + \delta K(0) = -\infty ,$$

contradicting (4). On the other hand, if $K_0 \leqq \bar{K}$, we have either $K'(0) = +\infty$ or $K'(0) = 0$, and the constraints are not violated. We can thus state, in general, that

(16) if $\bar{K}$ is the global maximum of $P(K)$, and if $K_0 \leqq \bar{K}$, then the optimal policy is $K(t) = \bar{K}$ for $t > 0$.[1]

Some observations implicit in the preceding discussion will be made explicit. Because of (1) and (4), $K(t)$ can have no downward jumps, though it can jump upward. It will be convenient to assume that $K(t)$ is continuous on the left; this assumption involves no loss of generality. The right-hand limit will be denoted, as usual, by $K(t+0)$. In this case, the effect of (2) is summed up in the condition

$$K(0+0) \geqq K_0 . \tag{17}$$

We eliminate I from (1) and (4) to obtain the constraint

$$K' + \delta K \geqq 0 . \tag{18}$$

To clarify the subsequent discussion, let us consider briefly the optimal policy with $K_0 > \bar{K}$ when $P(K)$ has only one local maximum, necessarily $\bar{K}$. In that case, for $K > \bar{K}$, $P(K)$ increases as K decreases. Hence optimal policy requires $K(t)$ to decrease as rapidly as possible until it reaches the value $\bar{K}$. The only restriction on its downward movement is (18), so that for an interval beginning at $t = 0$ we have $K' + \delta K = 0$, and therefore

$$K(t) = K_0 e^{-\delta t} .$$

At some time τ we shall find that $K(\tau) = \bar{K}$. At this point, the firm is in the same position, looking forward in time, as if $K_0 = \bar{K}$; hence the optimal solution is to continue with the solution $K(t) = \bar{K}$. We state

THEOREM 1. *If $P(K)$ has one and only one local maximum $\bar{K}$, then the optimal capital policy for given initial K_0 is*

(a) *if $K_0 \leqq \bar{K}$,*

$$K(t) = \bar{K} \qquad (t > 0) ;$$

[1] This result is somewhat paradoxical for both the capital policy and advertising policy problems, but more so for the latter. It says that if the initial stock of capital or good will is below the global maximum, the firm should invest or advertise at an infinitely rapid rate. While purchasing capital goods in a large block is not inconceivable, a huge, very brief advertising campaign to build up good will to the optimal level scarcely makes sense. The paradox arises from the unreal assumption that the stock of good will can be increased at a rate proportional to the rate of advertising. In fact, there would be diminishing returns as the increased expenditures force the advertising effort into increasingly unrewarding channels. Similarly, an infinitely rapid rate of investment might not be possible at constant prices, though this is not inconceivable if the firm is a relatively small part of the market for the capital goods.

(b) *if* $K_0 > K$,

$$K(t) = \begin{cases} K_0 e^{-dt} & 0 \leqq t \leqq \dfrac{\log_e K_0/\bar{K}}{\delta}, \\ \bar{K} & t > \dfrac{\log_e K_0/\bar{K}}{\delta}. \end{cases}$$

The argument leading to Theorem 1 need not be perfectly rigorous, since this theorem will later appear as a special case of Theorem 2. In the argument, we noted that choosing a policy to run from any time t on is the same as choosing a policy beginning at $t = 0$, where $K(t)$ is taken as the initial capital holding. This well-known recurrence principle for optimal sequential decision-making has been used by Wald [7, p. 105], Massé [5], and Arrow, Harris, and Marschak [1], and has been given general formulation by Bellman [3, p. 83] and Karlin [4]. In this case, we start from (13), which by an obvious change of variable can be written

$$\begin{aligned} W\{K(t)\} &= \int_0^{\tau} e^{-\alpha t} P[K(t)]\, dt + e^{-\alpha\tau} \int_0^{+\infty} e^{-\alpha t} P[K(t+\tau)]\, dt \\ &= \int_0^{\tau} e^{-\alpha t} P[K(t)]\, dt + e^{-\alpha\tau} W\{K(t+\tau)\} \end{aligned} \tag{19}$$

for any τ. Suppose that $K(t+\tau)$ is not an optimal policy with a beginning value $K(\tau)$. Define

$$K_1(t) = \begin{cases} K(t) & 0 \leqq t \leqq \tau, \\ \bar{K}(t+\tau) & t > \tau, \end{cases}$$

where $\bar{K}(t)$ is an optimal policy with initial capital $K(\tau)$. Then, by the definition of an optimal policy,

$$W\{\bar{K}(t)\} > W\{K(t+\tau)\},$$

and therefore

$$\begin{aligned} W\{K_1(t)\} &= \int_0^{\tau} e^{-\alpha t} P[K(t)]\, dt + e^{-\alpha\tau} W\{\bar{K}(t)\} \\ &> W\{K(t)\}, \end{aligned}$$

so that $W\{K(t)\}$ cannot be optimal.

(20) If $K(t)$ is an optimal policy with initial capital K_0, then for all $\tau > 0$ the policy $K(t+\tau)$ is optimal with initial capital $K(\tau)$.

By the same argument, we can show that

(21) if $\bar{K}_1(t)$ is an optimal policy and $\bar{K}_2(t)$ is optimal with initial capital $\bar{K}_1(\tau)$, the policy

$$\bar{K}_3(t) = \begin{cases} \bar{K}_1(t) & 0 \leqq t \leqq \tau \\ \bar{K}_2(t-\tau) & t > \tau \end{cases}$$

is also optimal.

From (21), we can deduce the following consequence:

LEMMA 1. *If $\bar{K}(t)$ is an optimal policy and $\bar{K}(\tau) = K_0$ for some $\tau > 0$, then the periodic policy*

$$K^*(t) = \bar{K}(t - j\tau) \qquad [j\tau \leqq t < (j+1)\tau]$$

is optimal. In particular, if $\bar{K}(t) = K_0$ for $0 \leqq t \leqq \tau$, where $\tau > 0$, then the stationary policy $K^*(t) \equiv K_0$ is optimal.

PROOF. Since at τ the initial capital is the same as at $t=0$, it would be optimal to continue with the policy begun at $t = 0$. Formally, let

$$K_1(t) \equiv \bar{K}(t) , \qquad K_2(t) = \begin{cases} K_1(t) & 0 \leqq t \leqq \tau , \\ K_1(t-\tau) & t > \tau . \end{cases}$$

Then $K_2(t)$ is also optimal by (21), and $K_2(2\tau) = K_1(\tau) = K_0$. Since $K_1(t)$ satisfies (17) and (18), clearly the same is true of $K_2(t)$. We may continue this process by defining recursively

$$K_{n+1}(t) = \begin{cases} K_n(t) & 0 \leqq t < n\tau , \\ K_n(t - n\tau) & t \geqq n\tau . \end{cases}$$

By induction we easily establish that $K_n(t)$ is optimal, and $K_n(n\tau) = K_0$. For any fixed t, the value of $K_n(t)$ is the same for all n sufficiently large, so that we may define the limit of this sequence of policies as

$$K^*(t) = \lim_{n \to \infty} K_n(t) ,$$

which obviously has the definition specified in the lemma. The policy $K^*(t)$ satisfies (17) and (18), and coincides with $K_n(t)$ for $t < n\tau$. Hence

$$\begin{aligned} | W\{K^*(t)\} - W\{K_n(t)\} | &= \left| \int_{n\tau}^{+\infty} e^{-\alpha t}\{P[K^*(t)] - P[K_n(t)]\}\, dt \right| \\ &\leqq e^{-\alpha n\tau} \int_0^{+\infty} e^{-\alpha t} \, | P[K^*(t + n\tau)] - P[K_n(t + n\tau)] | \, dt . \end{aligned}$$

Since $P(K)$ is a bounded function, the integral on the right is uniformly bounded, and therefore the right-hand side approaches zero as n approaches infinity. However, since $K_n(t)$ is optimal for all n, it follows that $W\{K_n(t)\}$ is the same for all n. Therefore

$$W\{K^*(t)\} = W\{K_n(t)\} ,$$

and $W\{K^*(t)\}$ is optimal.

3. Characterization of the Solution

We prove a series of lemmas which, taken together, characterize the optimal policy. The first case we investigate is the nature of the solution when the constraint (18) is not effective but there is no discontinuity.

LEMMA 2. (a) *If an optimal policy $\bar{K}(t)$ is differentiable at $t = t_0$, and $\bar{K}'(t_0) + \delta\bar{K}(t_0) > 0$, then $\bar{K}(t_0)$ is a local maximum or a local minimum of $P(K)$, and there is an interval containing t_0 for which $\bar{K}(t)$ is a constant.*

(b) *If* $\bar{K}(t)$ *is optimal, then* $\bar{K}'(t) \leqq 0$ *over every interval of differentiability.*
(c) *If the stationary policy* $\bar{K}(t) \equiv K_0$ *is optimal, then* K_0 *is a local maximum or a local minimum of* $P(K)$.

Actually, as we shall see (Lemma 4), $\bar{K}(t_0)$ will have to be a local maximum under the conditions of (a); similarly, K_0 in (c) will have to be a local maximum.

PROOF. (a) Suppose $\bar{K}(t_0)$ is neither a local maximum nor a local minimum. From the finiteness of the number of local maxima (and therefore of local minima), either $P(K)$ is increasing in an interval containing $\bar{K}(t_0)$ or it is decreasing in such an interval. We show by exhibiting another policy $K(t)$ with a higher surplus that the first case leads to a contradiction; the argument for the second case is completely parallel.

Choose t_1 greater than t_0 but sufficiently close to t_0 so that

$$\min_{t_0 \leqq t \leqq t_1} \bar{K}(t) \quad \text{and} \quad \exp[\delta(t_1 - t_0)]\bar{K}(t_1)$$

belong to the K-interval in which $P(K)$ is increasing. Define

$$K(t) = \begin{cases} \bar{K}(t) & 0 \leqq t \leqq t_0; \quad t > t_1\,, \\ \exp[\delta(t_1 - t)]\bar{K}(t_1) & t_0 < t \leqq t_1\,. \end{cases}$$

First, we show that $K(t) > \bar{K}(t)$ for $t_0 < t < t_1$. In what follows, $t_0 < t < t_1$ except where otherwise noted. From the hypothesis, by choosing t_1 sufficiently close to t_0 we can ensure that $\bar{K}'(t) + \delta\bar{K}(t) > 0$. On the other hand, $K'(t) + \delta K(t) = 0$. Let $k(t) = K(t) - \bar{K}(t)$. Then $k'(t) + \delta k(t) < 0$, or $\{d[e^{\delta t}k(t)]\}/dt < 0$, so that

$$e^{\delta t}k(t) > e^{\delta t_1}k(t_1) = 0\,,$$

since $K(t_1) = \bar{K}(t_1)$. Therefore, $k(t) > 0$, or $K(t) > \bar{K}(t)$. Now,

$$\exp[\delta(t_1 - t_0)]\bar{K}(t_1) > K(t) > \bar{K}(t) \geqq \min_{t_0 \leqq t \leqq t_1} \bar{K}(t)\,,$$

so that for each t, the policies $K(t)$ and $\bar{K}(t)$ both lie in an interval where $P(K)$ is increasing, and

$$P[K(t)] > P[\bar{K}(t)] \qquad (t_0 < t < t_1)\,.$$

Since $P[K(t)] = P[\bar{K}(t)]$ for all other t, we have $W\{K(t)\} > W\{\bar{K}(t)\}$, which is, as noted, a contradiction.

To complete the proof of (a), suppose that $\bar{K}'(t_0) + \delta\bar{K}(t_0) > 0$ at some t_0. Then $\bar{K}'(t) + \delta\bar{K}(t) > 0$ in some interval containing t_0. At each point of this interval, $\bar{K}(t)$ is a local extremum, as just shown. Since there are only a finite number of such local extrema and since $\bar{K}(t)$ is differentiable in this interval, it must be constant there.

(b) For any t for which $\bar{K}(t)$ is differentiable, either $\bar{K}'(t) + \delta\bar{K}(t) > 0$ or $\bar{K}'(t) + \delta\bar{K}(t) = 0$, from (18). In the first case we have $\bar{K}'(t) = 0$, from (a); in the second, $\bar{K}'(t) = -\delta\bar{K}(t) \leqq 0$.

(c) If $\bar{K}(t)$ is constant, then $\bar{K}'(t) \equiv 0$ and $\bar{K}'(t) + \delta\bar{K}(t_0) \equiv \delta K_0 > 0$, so that (a) applies.

As we have seen in section 2, a key role in the solution is played by policies that have a discontinuity at the origin and are constant thereafter. The jump is motivated by the desire to achieve a higher value of $P(K)$; therefore the constant value should be as high a maximum as possible. We give the following formal definition:

(22) A policy $K(t)$ is termed the *jump policy* relative to K_1 (with initial capital K_0) if $K(t) = \bar{K} > K_0$ for $t > 0$, where $\bar{K}$ uniquely maximizes $P(K)$ subject to $K \geqq K_1$.

The jump policy is uniquely defined if there exists a $\bar{K} > K_0$ that uniquely maximizes $P(K)$ subject to $K \geqq K_1$; otherwise it is not defined. The local maxima of $P(K)$ subject to $K \geqq K_1$ are those local maxima of $P(K)$ that are greater than K_1, plus possibly K_1 itself; in view of (8), the global maximum $\bar{K}$ will fail of uniqueness only if $P(K_1)$ equals the value of $P(K)$ at the highest local maximum above K_1, and we wish to exclude this case. The condition $\bar{K} > K_0$ ensures that (17) is satisfied; the strict inequality ensures that there will actually be a jump. Finally, since $K' = 0$ for all $t > 0$ on a jump policy, (18) is satisfied.

The reason for distinguishing between K_0 and K_1 will become clearer below (see the proof of Lemma 6). We shall be interested in the case $K_1 \leqq K_0$, and it is easily seen, in view of (17), that a jump policy which is optimal for $K_0 = K_1$ remains optimal as K_0 increases but remains below $\bar{K}$.

LEMMA 3. *Let $K_1(t)$ be the jump policy relative to K_1 and $K(t)$ be any other policy satisfying* (17) *and* (18) *for which $K(t) \geqq K_1$ for all $t > 0$. Then $W\{K_1(t)\} > W\{K(t)\}$.*

PROOF. By construction,

$$P(\bar{K}) \geqq P[K(t)] \qquad (t > 0)\,. \tag{23}$$

Since $K(t)$ is not the jump policy, $K(t_0) \neq \bar{K}$ for some $t_0 > 0$. If $K(t_0) < \bar{K}$, it is impossible that $K(t) \geqq \bar{K}$ in every left-hand interval of t_0, for otherwise $\bar{K}'(t_0)$ would equal $-\infty$, contradicting (18). Hence there is an interval on which $K(t) < \bar{K}$. Similarly, if $K(t_0) > \bar{K}$, there is an interval on which $K(t) > \bar{K}$, and in either case there is an interval on which $K(t) \neq \bar{K}$. Since $K(t) \geqq K_1$, it follows from (22) that the strict inequality holds in (23) on some interval. From the definition (13) of W, the lemma follows.

We can now sharpen Lemma 2.

LEMMA 4. (a) *If K_0 is a stationary optimal policy, then K_0 is a local maximum of $P(K)$, $P(K) < P(K_0)$ for $K > K_0$, and there is no jump policy relative to K_0.* (b) *If $\bar{K}(t)$ is optimal and is differentiable at $t = t_0$ and if*

$$\bar{K}'(t_0) + \delta\bar{K}(t_0) > 0\,,$$

then $\bar{K}(t_0)$ is a stationary optimal policy.

PROOF. (a) If there exists a jump policy relative to K_0, then by Lemma 3 it must be better than the stationary policy. If, then, the stationary policy is optimal, there can exist no jump policy. If $P(K) > P(K_0)$ for some $K > K_0$ and if $\bar{K}$ maximizes $P(K)$ for $K \geqq K_0$, then $\bar{K} > K_0$ and $\bar{K}$ must be unique, as already observed; therefore, there would be a jump policy. Hence $P(K) \leqq P(K_0)$ for $K \geqq K_0$; that is, K_0 is a right-hand maximum of $P(K)$. If K_0 were a local minimum, then $P(K)$ would have to be constant in a right-hand neighborhood of K_0, so there would be a continuum of local maxima, contrary to (7). From Lemma 2(c), K_0 then has to be a local maximum of $P(K)$. Suppose that $P(K_1) = P(K_0)$ for some $K_1 > K_0$. Then $P(K_1) \geqq P(K)$ for all $K \geqq K_0$, so that K_1 would be a local maximum of $P(K)$. But then there would be two local maxima of $P(K)$ with the same value, contradicting (8); hence $P(K_0) > P(K)$ for all $K > K_0$.

(b) If $\bar{K}(t)$ is optimal, then by (20) the policy $\bar{K}(t + t_0)$ is optimal for initial capital $\bar{K}(t_0)$. But $\bar{K}(t)$ is constant in any interval containing t_0 by Lemma 2(a), and therefore $\bar{K}(t + t_0)$ is constant in a right-hand interval of 0. By Lemma 1, the stationary policy $\bar{K}(t_0)$ is also optimal.

We now confirm the usefulness of our definition (22) of a jump policy by observing that if a policy with a jump at $t = 0$ is optimal, it must be a jump policy.

Lemma 5. *If $\bar{K}(t)$ is optimal and has a jump at $t = 0$, then it is the jump policy relative to K_0.*

Proof. The argument will be that if $\bar{K}(t)$ is an optimal policy but not a jump policy, it is possible to construct an optimal policy that never falls below K_0 and that is not a jump policy. But this will contradict Lemma 3.

If $\bar{K}(t) \geqq K_0$ for all t, let $K^*(t) = \bar{K}(t)$. Otherwise, $\bar{K}(t) < K_0$ for some t_0. Let

$$\tau = \inf \{t : \bar{K}(t) < K_0\} .$$

Since $\bar{K}(t)$ has a jump at $t = 0$, there is a right-hand interval of 0 in which $K(t) \geqq K_0$;

$$\tau > 0, \quad \bar{K}(t) \geqq K_0 \qquad (0 \leqq t \leqq \tau) . \tag{24}$$

Since $\bar{K}(t)$ can have no downward jumps, we must have $\bar{K}(\tau) = K_0$. By Lemma 1, the periodic policy

$$K^*(t) = \bar{K}(t - j\tau) \qquad [j\tau \leqq t < (j + 1)\tau]$$

is also optimal, while from (24) we have $K^*(t) \geqq K_0$ for all t. Also, $K^*(j\tau) = \bar{K}(0) = K_0$ for all j, where $K^*(t) = \bar{K}(t) > K_0$ for t sufficiently small (since there is a jump at $t = 0$). Hence $K^*(t)$ is not constant for $t > 0$, and therefore $K^*(t)$ is not a jump policy.

We have shown that if there is an optimal policy with a jump at $t = 0$ that is not a jump policy, there is an optimal policy $K^*(t)$ with a jump at $t = 0$ which is not a jump policy but for which $K^*(t) \geqq K_0$ for all t. If we can show that there exists a jump policy, the conclusion will follow from Lemma 3.

If $P(K) > P(K_0)$ for some $K > K_0$, then the global maximum $\bar{K}$ of $P(K)$

subject to $K \geqq K_0$ cannot equal K_0 and must be unique, as seen earlier, and the jump policy exists. Suppose $P(K) \leqq P(K_0)$ for all $K \geqq K_0$. Then

$$P(K_0) \geqq P[K^*(t)] \quad \text{for all } t; \tag{25}$$

since $K^*(t)$ is optimal, so is the stationary policy K_0. By Lemma 4(a), $P(K_0) > P(K)$ for all $K > K_0$. Since $K^*(t) > K_0$ for t sufficiently small, the strict inequality holds in (25) for some t-interval, which contradicts the optimality of $K^*(t)$. Hence a jump policy must exist and be optimal.

LEMMA 6. *If $\bar{K}(t)$ is optimal and has a jump anywhere, it must be the jump policy relative to K_0.*

PROOF. Let t_0 be the greatest lower bound of the jumps. Then for any jump t_1, the policy $\bar{K}(t + t_1)$ is optimal for initial capital $\bar{K}(t_1)$ by (20), and has a jump at $t = 0$, so that by Lemma 5 it must be a jump policy and therefore constant for $t > t_1$. Since t_1 can be chosen arbitrarily close to t_0, it follows that $\bar{K}(t)$ must be constant for $t > t_0$, and the only jump occurs at t_0. We wish to show that $t_0 = 0$, in which case the result follows from Lemma 5.

Suppose $t_0 > 0$. Then $\bar{K}(t)$ is differentiable for $0 \leqq t \leqq t_0$ and by Lemma 2(b) is monotone-decreasing there. If $\bar{K}(t) = \bar{K} > \bar{K}(t_0)$ for $t > t_0$, we can choose $t_2 < t_0$ so that $\bar{K}(t_0) \leqq \bar{K}(t_2) < \bar{K}$. Define

$$K(t) = \begin{cases} \bar{K}(t_2) & t = 0\,, \\ \bar{K} & t > 0\,. \end{cases}$$

Since $\bar{K}(t + t_0)$ is optimal for $\bar{K}(t_0)$ and has a jump at $t = 0$, by Lemma 5 it must be a jump policy relative to $\bar{K}(t_0)$; and by definition (22), $\bar{K} > \bar{K}(t_0)$ maximizes $P(K)$ uniquely, subject to $K \geqq \bar{K}(t_0)$. Since $\bar{K}(t_2) < \bar{K}$, it follows from (22) that $K(t)$ is the jump policy relative to $\bar{K}(t_0)$ with initial capital $\bar{K}(t_2)$. But $\bar{K}(t + t_2)$ satisfies (17) and (18) and the condition $\bar{K}(t + t_2) \geqq \bar{K}(t_0)$, since $\bar{K}(t + t_0)$ is monotone-decreasing for $0 \leqq t \leqq t_0 - t_2$ and is equal to $\bar{K} > \bar{K}(t_0)$ for $t > t_0 - t_2$. By Lemma 3,

$$W\{K(t)\} > W\{\bar{K}(t + t_2)\}\,,$$

which is impossible, since $\bar{K}(t + t_2)$ is optimal for initial capital $\bar{K}(t_2)$ by (20). The assumption $t_0 > 0$ has led to a contradiction.

4. Detailed Structure of the Optimal Policy

With preceding lemmas, we can now describe the structure of the optimal policy for any initial capital K_0. First consider an optimal policy that has no jumps. From (18) we have $\bar{K}' + \delta\bar{K} \geqq 0$ everywhere, while from Lemma 2(b) we have $\bar{K}' \leqq 0$ everywhere. If the strict inequality holds in (18), by Lemma 2(a) there is an interval of constancy, by Lemma 4(b) the constant value of $K(t)$ must be a local maximum of $P(K)$, and there are only a finite number of such local maxima. Between the intervals of constancy we have intervals for which the equality holds in (18); these are the *zero-investment*