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I. Moerdijk

Classifying Spaces and Classifying Topoi

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Preface

In these notes, a detailed account is presented of the relation between classifying spaces and classifying topoi. To make the notes more accessible, I have tried to keep the prerequisites to a minimum, for example by starting with an introductory chapter on topos theory, and by reviewing the necessary basic properties of geometric realization and classifying spaces in the first part of Chapter III. Furthermore, I have made an attempt to present the material in such a way that it is possible to read the special case of discrete categories first. This case already provides a good general picture, while it avoids some of the technical complications involved in the general case of topological categories. Thus, to reach the comparison and classification theorems for discrete categories in Section IV.1, the reader can omit §§3,4,5,7 and most of §6 in Chapter II, as well as the second parts of §1 and §2 in Chapter III.

In the past several years I have been helped by discussions with several people which were directly or indirectly related to the subject matter of these notes. In this respect, I am particularly indebted to W.T. van Est, S. Mac Lane, G. Segal and J.A. Svensson. Above all, A. Joyal taught me not to underestimate the Sierpinski space.

A summary of the main results appeared in the Comptes Rendus de l'Académie des Sciences (t. 317, 1993). The present version was mainly written during the fall of 1994, which I spent at the University of Aarhus. I am most grateful for the hospitality and support of the mathematical institute there. I would also like to thank A. Dold for the possibility to publish these notes in the Springer Lecture Notes Series, and for some advice on exposition. Finally, I would like to thank Elise Goeree for her careful typing of the manuscript.

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Utrecht, Spring 1995.

Contents

Introduction

These notes arose out of two related questions. First, what does the so-called classifying space of a small category actually classify? And secondly, what is the relation between classifying spaces and classifying topoi?

These questions can perhaps best be explained by describing the well-known case of a group G. The classifying space BG classifies principal G-bundles (or covering spaces with group G), in the sense that for any suitable space X (e.g., a CW-complex) there is a bijective correspondence between isomorphism classes of such covering projections $E \to X$ and homotopy classes of maps $X \to BG$. Furthermore, the cohomology groups of this space BG are exactly the Eilenberg-Mac Lane cohomology groups of the group G.

On the other hand, there is the classifying topos of the group G, introduced by Grothendieck and Verdier in SGA4, and defined as the category of all sets equipped with an action by the group G. I will denote this category by $\mathcal{B}G$. The topos $\mathcal{B}G$ has the same properties as the space BG, for tautological reasons: the cohomology of the topos $\mathcal{B}G$ is the group cohomology of G, because the definitions of topos cohomology and group cohomology are verbally the same in this case. And for any other topos $\mathcal{T}$, the fact that topos maps from $\mathcal{T}$ into $\mathcal{B}G$ correspond to principal G-bundles over $\mathcal{T}$ is an elementary consequence of the definition of a map between topoi.

To compare the classifying space BG and the classifying topos $\mathcal{B}G$ of G-sets, one first has to put these two objects in one and the same category. For this reason, we replace the space BG by its topos $Sh(BG)$ of all sheaves (of sets) on BG.

More generally, it will be explained in Chapter I how for any space X, the topos $Sh(X)$ of sheaves on X contains basically the same information as the space X itself, and should be viewed simply as the space X disguised as a topos. This view is supported by the fact that for two spaces X and Y, continuous mappings between X and Y correspond to topos mappings between $Sh(X)$ and $Sh(Y)$. Moreover, for a sufficiently good space X (e.g., a CW-complex), the cohomology groups of the space X are the same as those of the topos $Sh(X)$.

To come back to the comparison between the space BG and the topos $\mathcal{B}G$ of G-sets, we note that after having replaced BG by its topos $Sh(BG)$, the two can be related by a mapping $Sh(BG) \to \mathcal{B}G$. This topos map is a weak homotopy equivalence, although $\mathcal{B}G$ is a much smaller and simpler topos than $Sh(BG)$. The known isomorphisms between the cohomology and homotopy groups of the space BG and those of the topos $\mathcal{B}G$ are induced by this map $Sh(BG) \to \mathcal{B}G$. Furthermore, it follows that for a CW-complex X, there is a bijective correspondence between homotopy classes of maps between spaces $X \to BG$ and homotopy classes of topos maps $Sh(X) \to \mathcal{B}G$. In this way, the fact that the space BG classifies principal G-bundles can be seen as a consequence of the fact that the topos $\mathcal{B}G$ does.

The first purpose in these notes will be to extend this relation between classifying space and classifying topos from the well-known and elementary case of a group G to that of an arbitrary small category $\mathbf{C}$. In Chapter I, we will recall how the classifying topos $\mathcal{B}\mathbf{C}$ of $\mathbf{C}$ is constructed as the topos of all presheaves on $\mathbf{C}$, i.e. of all contravariant set-valued functors on $\mathbf{C}$. In Chapter III, §2, it will be recalled how the classifying space $B\mathbf{C}$ is constructed as the geometric realization of the nerve of $\mathbf{C}$. As for groups, it is known that the two constructions define the same cohomology (for locally constant, abelian coefficients). We will relate the two constructions, by first replacing the space $B\mathbf{C}$ by its topos $Sh(B\mathbf{C})$, and then constructing a weak homotopy equivalence of topoi (see Theorem 1.1 in Chapter IV):

$$p : Sh(B\mathbf{C}) \longrightarrow \mathcal{B}\mathbf{C}. \tag{1}$$

The construction of this map p is based on a comparison of various types of geometric realization, for spaces as well as for topoi and using different kinds of intervals, to be presented in Chapter III.

Of course, a lot more information is contained in a weak homotopy equivalence (1) than in the mere fact that the space $B\mathbf{C}$ and the topos $\mathcal{B}\mathbf{C}$ have isomorphic cohomology groups. For example, from the existence of such a map $Sh(B\mathbf{C}) \to \mathcal{B}\mathbf{C}$, one can conclude that for any CW-complex X there is a bijective correspondence between homotopy classes of maps of spaces $X \to B\mathbf{C}$ and homotopy classes of maps of topoi $Sh(X) \to \mathcal{B}\mathbf{C}$:

$$[X, B\mathbf{C}] = [Sh(X), \mathcal{B}\mathbf{C}]. \tag{2}$$

Using this bijective correspondence, one can transfer known classification results for the topos $\mathcal{B}\mathbf{C}$ to the space $B\mathbf{C}$. Indeed, define a principal $\mathbf{C}$-bundle E on a space X to be a system of sheaves $E(c)$, one for each object c in $\mathbf{C}$, on which $\mathbf{C}$ acts by sheaf maps $\alpha_* : E(c) \to E(d)$ for each arrow $\alpha : c \to d$ in $\mathbf{C}$, in a functorial way. Moreover, the bundle E should satisfy the following three conditions for being principal, for each point x in X (where $E(c)_x$ denotes the stalk of $E(c)$ at x):

(i) $\bigcup_{c \in \mathbf{C}} E(c)_x$ is non-empty.

(ii) The action is transitive: given $y \in E(c)_x$ and $z \in E(d)_x$, there are arrows $\alpha : b \to c$ and $\beta : b \to d$ in $\mathbf{C}$ and a point $w \in E(b)_x$ for which $\alpha_*(w) = y$ and $\beta_*(w) = z$.

(iii) The action is free: given $y \in E(c)_*$ and parallel arrows $\alpha, \beta : c \rightrightarrows d$ in $\mathbf{C}$ so that $\alpha_*(y) = \beta_*(y)$, there exists an arrow $\gamma : b \to c$ in $\mathbf{C}$ and a point $z \in E(b)_x$, for which $\alpha\gamma = \beta\gamma$ and $\gamma_*(z) = y$.

Note that in case $\mathbf{C}$ is a group (viewed as a category with only one object), this definition of principal bundle agrees with the usual one.

A basic result of topos theory, which we will review in Section II.2, states that there is an exact correspondence between such principal $\mathbf{C}$-bundles over X and topos maps $Sh(X) \to \mathcal{B}\mathbf{C}$. Using this correspondence and the bijection (2) above, one obtains for a CW-complex X and a small category $\mathbf{C}$ the following theorem, to be proved in Section IV.1:

Theorem. *Homotopy classes of maps $X \to B\mathbf{C}$ are in bijective correspondence with concordance classes of principal* $\mathbf{C}$*-bundles over X.*

Here two principal bundles over X are said to be concordant if they lie at the two ends of some principal bundle over $X \times [0,1]$.

This theorem of course contains the classical fact that the classifying space BG of a group G classifies principal G-bundles. The theorem also extends a result of G. Segal, which states that for a monoid with cancellation M, its classifying space BM classifies a suitably defined notion of principal M-bundle.

Thus, the weak equivalence (1) and the theorem above together provide an answer to the two questions stated at the beginning of this introduction, for the case of a discrete category $\mathbf{C}$.

Much of the work in these notes is concerned with the problem of extending these results to topological categories. Recall that a topological category $\mathbf{C}$ is given by a space of objects $\mathbf{C}_0$ and a space of arrows $\mathbf{C}_1$, together with continuous operations for source and target $\mathbf{C}_1 \rightrightarrows \mathbf{C}_0$, for identity arrows $\mathbf{C}_0 \to \mathbf{C}_1$, and for composition $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \to \mathbf{C}_1$. For example, any topological group or monoid is a topological category (with a space of objects which consists of just one point), as is any topological groupoid, such as the holonomy groupoid of a foliation (Haefliger(1984), Bott(1972), Segal(1968)). The construction of the classifying topos of a topological category will be described in detail in §II.3 while the classical construction of the classifying space will be reviewed in §III.2. The general considerations concerning geometric realization will again provide a map as in (1) relating the classifying space and the classifying topos. This map will in general not be a weak homotopy equivalence. However, there is an interesting case, which includes that of discrete categories, where the map is a weak homotopy equivalence. This is the case of topological categories $\mathbf{C}$ with the property that their source map $s : \mathbf{C}_1 \to \mathbf{C}_0$ is étale, i.e. is a local homeomorphism. Such topological categories will be called s-étale. For example, many of the topological groupoids arising in the theory of foliations are s-étale, as are topological categories constructed from diagrams of spaces (see §II.5 below). For s-étale topological categories, the map (1) is a weak homotopy equivalence, as said; moreover, it will be shown in §II.4 that the correspondence between topos maps into $\mathcal{B}\mathbf{C}$ and principal $\mathbf{C}$-bundles, already referred to above, generalizes to the case of s-étale topological categories $\mathbf{C}$. It follows that the theorem just stated also holds for s-étale topological categories.

As an illustration of the use of classifying topoi for discrete and s-étale categories, we will present in §IV.3 a relatively straightforward topos theoretic proof of Segal's theorem on the weak homotopy type of the Haefliger groupoid Γ^q.

For an arbitrary topological groupoid (not necessarily s-étale) the "naively" constructed classifying topos $\mathcal{B}\mathbb{C}$ need not contain much information. To obtain a suitable comparison with the classifying space $B\mathbb{C}$, we will consider a different classifying topos for $\mathbb{C}$, described by Deligne. Recall that in Deligne(1975), the notion of a sheaf on a simplicial space Y is introduced, and the topos $Sh(Y)$ (Deligne writes $\tilde{Y}$) of all such sheaves is considered as an alternative for the geometric realization $|Y|$. In particular for a topological category $\mathbb{C}$, and its associated simplicial space Nerve($\mathbb{C}$), the topos Sh(Nerve($\mathbb{C}$)) provides an alternating for the classifying space $B\mathbb{C}$. This topos Sh(Nerve($\mathbb{C}$)) will be called the Deligne classifying topos of $\mathbb{C}$, and be denoted by $\mathcal{D}\mathbb{C}$.

Deligne shows in op. cit. that for a suitable simplicial space Y the realization and the topos of sheaves $Sh(Y)$ have isomorphic cohomology groups. In §IV.4 it will be shown that this isomorphism in cohomology is induced by a map, and that the topos $Sh(Y)$ has the same weak homotopy type as the geometric realization $|Y|$. In particular, this will show that for any topological category $\mathbb{C}$, the Deligne classifying topos $\mathcal{D}\mathbb{C}$ and the classifying space $B\mathbb{C}$ have the same weak homotopy type. From this last result, one can obtain an answer to the question what $B\mathbb{C}$ classifies: it will be shown that homotopy classes of maps $X \to B\mathbb{C}$ correspond to concordance classes of sheaves of linear orders on X equipped with a suitable augmentation into the category $\mathbb{C}$.

These notes by no means provide a complete picture of the comparison between classifying spaces and classifying topoi for topological categories, and many questions remain. One obvious question for the case of a topological group(oid) G is the precise relation between linear orders augmented by G which are shown to be classified by BG in these notes, and principal G-bundles. Another question concerns the relationship between the "small" classifying topoi $\mathcal{B}\mathbb{C}$ and $\mathcal{D}\mathbb{C}$ of a topological category, and the classifying "gros" topoi defined over the topological gros topos by Grothendieck and Verdier (see e.g. SGA4 (tome 1), p.317)).

Chapter I

Background in Topos Theory

§1 Basic definitions

A topos is a "generalized" topological space. Indeed according to Grothendieck, topoi (should) form the proper subject of study for topology. The basic idea is similar to that of various well-known dualities. For example, Gelfand duality states that one could replace a compact Hausdorff space X by its ring $C(X)$ of complex-valued functions; mappings between such spaces can be described in terms of these rings, and the space X can be recovered (up to homeomorphism) from $C(X)$.

Similarly, one can use the "ring" (category) of sets instead of the ring of complex numbers, and replace a space X by the collection of all its "continuous set-valued functions"; i.e. the sheaves of sets on X, described in detail in the next section. As for Gelfand duality, mappings between spaces can be described in terms of these sheaves, and the space X can be recovered from the collection of all its sheaves.

The definition of a topos is meant to capture the basic properties of this category of all sheaves on a space X, and similar categories. Sheaves of sets are taken as basic here, since abelian sheaves, simplicial sheaves, etc., can all be defined in terms of sheaves of sets. We present the definition of a topos in the "Giraud form", which requires some elementary categorical notions to be explained first. (For background in category theory, Chapters I - IV of Mac Lane(1971) suffice.)

Let $\mathcal{E}$ be a category. (It is our convention that the objects of $\mathcal{E}$ can form a proper class, but that for any two objects A and B the collection $\mathrm{Hom}(A, B)$ of all arrows from A to B is a set. If the objects of $\mathcal{E}$ form a set as well, $\mathcal{E}$ is said to be "small".)

1.1. Definition. A category $\mathcal{E}$ is said to be a *topos* iff it satisfies the Giraud axioms (G1-G4), to be stated below.

(G1) The category $\mathcal{E}$ has finite limits.

This axiom needs no further explanation. For the second axiom, we recall that a sum (coproduct) $\sum_{i \in I} E_i$, indexed by some set I, is said to be *disjoint* if for any two distinct indices j and k the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & E_k \\ \downarrow & & \downarrow \\ E_j & \longrightarrow & \sum E_i \end{array}$$

is a pullback; here the maps into the coproduct are the canonical ones, and 0 denotes the initial object of $\mathcal{E}$ (this is the sum of the empty family). If each E_i is equipped with an arrow $E_i \to A$ into a given object A, then the sum also has such an evident arrow $\sum E_i \to A$. Thus for any map $B \to A$, there is a canonical map

$$\sum B \times_A E_i \quad \to \quad B \times_A \sum E_i \,. \tag{1}$$

Sums in $\mathcal{E}$ are said to be *stable* if this map (1) is always an isomorphism – in other words, if sums commute with pullbacks. The second Giraud axiom now is:

(G2) All (set-indexed) sums exist in $\mathcal{E}$, and are disjoint and stable.

For the next axiom, consider an object E in $\mathcal{E}$ and a monomorphism $r : R \rightarrowtail E \times E$. For any object T in $\mathcal{E}$, composition with r defines a subset

$$\mathrm{Hom}(T, R) \subseteq \mathrm{Hom}(T, E \times E) \cong \mathrm{Hom}(T, E) \times \mathrm{Hom}(T, E).$$

If, for every object T, this subset is an equivalence relation on the set $\mathrm{Hom}(T, E)$, then the monomorphism $r : R \rightarrowtail E \times E$ is said to be an *equivalence relation* on E. For example, if $f : E \to F$ is any arrow, then the pullback $E \times_F E \rightarrowtail E \times E$ is an equivalence relation on E. A diagram

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} E \overset{f}{\to} F$$

in $\mathcal{E}$ is said to be *exact* if f is the coequalizer of r_1 and r_2 and

$$\begin{array}{ccc} R & \xrightarrow{r_2} & E \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[f]{} & F \end{array}$$

is a pullback. It is said to be *stably exact* if for any arrows $F \to A \leftarrow B$, the diagram

$$B \times_A R \;\rightrightarrows\; B \times_A E \;\to\; B \times_A F\,,$$

obtained by pullback along $B \to A$, is again exact. The third Giraud axiom is:

(G3) (a) For every epimorphism $E \to F$ in $\mathcal{E}$, the diagram $E \times_F E \rightrightarrows E \to F$ is stably exact.

(b) For every equivalence relation $R \rightarrowtail E \times E$, there exists an object E/R which fits into an exact diagram $R \rightrightarrows E \to E/R$.

It follows that any exact diagram in $\mathcal{E}$ is stably exact. It also follows that all small colimits exist in the category $\mathcal{E}$, since these can be constructed from sums and coequalizers of equivalence relations (as in G3 (b)); see Mac Lane - Moerdijk (1992), p. 577.

For the last axiom, recall that a collection of objects $\{G_i : i \in I\}$ of $\mathcal{E}$ is said to *generate* $\mathcal{E}$ when for any two parallel arrows $u, v : E \rightrightarrows F$ in $\mathcal{E}$, if $u \circ t = v \circ t$ for every arrow $t : G_i \to E$ from every G_i, then $u = v$. The collection $\{G_i : i \in I\}$ of objects is said to be *small* if it is a set (rather than a proper class).

(G4) The category $\mathcal{E}$ has a small collection of generators.

If $\{G_i : i \in I\}$ is a set of generators, then every object E in $\mathcal{E}$ is a colimit of such generating objects.

A *morphism* between topoi $f : \mathcal{F} \to \mathcal{E}$ consists of a pair of functors ("inverse" and "direct" image functors)

$$f^* : \mathcal{E} \to \mathcal{F} \text{ and } f_* : \mathcal{F} \to \mathcal{E}$$

with the following two properties:

(i) f^* is left adjoint to f_* ; i.e. there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{F}}(f^*E, F) \cong \mathrm{Hom}_{\mathcal{E}}(E, f_*F),$$

(ii) f^* commutes with finite limits (i.e., is "left exact").

Such morphisms $f : \mathcal{F} \to \mathcal{E}$ and $g : \mathcal{G} \to \mathcal{F}$ can be composed in the evident way,

$$(f \circ g)^* = g^* \circ f^* \ , \ (f \circ g)_* = f_* \circ g_* \ .$$

Since the inverse image f^* of any morphism f is a left adjoint, it commutes with colimits. Therefore, since every object of $\mathcal{E}$ is a colimit of generators, f^* is completely determined (up to natural isomorphism) by its behaviour on generators. Furthermore, any functor $f^* : \mathcal{E} \to \mathcal{F}$ which commutes with colimits must have a right adjoint, necessarily unique up to isomorphism (Mac Lane (1971), p. 83). Thus topos morphisms can be described more economically, and we see will some explicit examples of this later.

The collection of all morphisms $f : \mathcal{F} \to \mathcal{E}$ has itself the structure of a category: an arrow δ between two morphisms $f, g : \mathcal{F} \to \mathcal{E}$ is a natural transformation

$$\delta : f^* \to g^*$$

between the inverse image functors. By the remarks above, this category $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$ is equivalent to the category of functors $f^* : \mathcal{E} \to \mathcal{F}$ which commute with colimits and finite limits, and natural transformations between them.

A morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be an *equivalence* if there exists a morphism $g : \mathcal{E} \to \mathcal{F}$ and isomorphisms $f \circ g \cong \mathrm{id}_{\mathcal{E}}$ and $g \circ f \cong \mathrm{id}_{\mathcal{F}}$. This is equivalent to the requirement that the unit $id_{\mathcal{E}} \to f_* f^*$ and the counit $f^* f_* \to id_{\mathcal{F}}$ are natural isomorphisms. The topoi $\mathcal{E}$ and $\mathcal{F}$ are said to be equivalent if there exists such an equivalence f, and one writes

$$\mathcal{E} \cong \mathcal{F}$$

in this case. In practice, one often tacitly identifies equivalent topoi, just as one identifies homeomorphic spaces. However, given two topoi $\mathcal{E}$ and $\mathcal{F}$, one cannot always identify isomorphic morphisms $\mathcal{F} \to \mathcal{E}$, as will be clear, e.g. from the discussion of pushouts of topoi in Section 3.

If $\mathcal{E}$ is a topos and B is an object in $\mathcal{E}$, one can form the "comma-category" $\mathcal{E}/B$, with as objects the arrows $E \to B$ in $\mathcal{E}$, and as arrows in $\mathcal{E}/B$ the commutive triangles in $\mathcal{E}$. This category $\mathcal{E}/B$ again satisfies the Giraud axioms for a topos: it inherits all the required exactness properties from $\mathcal{E}$; and if $\{G_i : i \in I\}$ is a set of generators for $\mathcal{E}$, then the collection of all arrows $G_i \to B$ (for all $i \in I$) is a set of generators for $\mathcal{E}$. The functor $E \mapsto (\pi_2 : E \times B \to B) : \mathcal{E} \to \mathcal{E}/B$ commutes with colimits and finite limits, and hence is the inverse image functor of a topos morphism $\mathcal{E}/B \to \mathcal{E}$. (Its direct image part Π_B is described explicitly, e.g. in Mac Lane-Moerdijk (1992),p. 60.)

§2 First examples

In this section we describe the topos of sheaves on a space and the topos of presheaves on a small category. Before doing so, we should mention the simplest example of a topos, viz. the category of all small sets, denoted $\mathcal{S}$ or (*sets*). (One readily verifies the Giraud axioms (G1-4) for $\mathcal{S}$; for a collection of generators, one can take the one-element collection consisting of the one-point set.)

For any other topos $\mathcal{E}$, there is a morphism $\gamma : \mathcal{E} \to \mathcal{S}$, unique up to isomorphism. It can be described explicitly, in terms of the terminal object 1 of $\mathcal{E}$, by

$$\gamma^*(S) = \sum_{s \in S} 1 \ , \quad \gamma_*(E) = \mathrm{Hom}_{\mathcal{E}}(1, E) \ ,$$

for any set S and any object E in $\mathcal{E}$. One often writes Δ for γ^* and Γ for γ_*. The functor Δ is called the *constant sheaf functor*, and Γ the *global sections functor*.

Now let X be a topological space. A continuous map $f : E \to X$ is said to be a *local homeomorphism* (or, an *étale map*, or an *étale space* over X) if both f and its diagonal $E \to E \times_X E$ are open maps. This is equivalent to the requirement that for any point $y \in E$ there exist open neighbourhoods $V_y \subseteq E$ and $U_{f(y)} \subseteq X$ so that f restricts to a homeomorphism $f : V_y \xrightarrow{\sim} U_{f(y)}$. A *sheaf on* X is such an étale map $f : E \to X$. A map φ between sheaves $(f : E \to X) \to (f' : E' \to X)$ is a continuous map $\varphi : E \to E'$ so that $f' \circ \varphi = f$. This defines a category of all sheaves on X, denoted

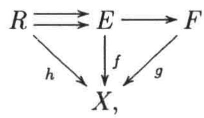
$$Sh(X)\ .$$

[In the literature, one often defines a sheaf (of sets) as a functor $F : \mathcal{O}(X)^{op} \to (sets)$, defined on the poset $\mathcal{O}(X)$ of all open subsets of X, and having for each open cover $U = \bigcup U_i$ the "unique pasting property" that the diagram

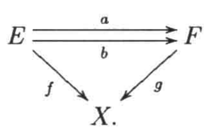
$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer of sets. These definitions are of course equivalent, as is explained in any book on sheaf theory; see e.g. Godement(1958), Swan(1964).]

The category $Sh(X)$ is a topos. Indeed, finite limits and colimits are constructed just as for topological spaces, because these constructions preserve étale maps. More explicitly, if $E \to X$ and $F \to X$ are two étale maps then so are $E \times_X F \to X$ and $E + F \to X$, and these represent the product and sum in the category $Sh(X)$. The same applies to infinite sums. Similarly, in an exact diagram of topological spaces over X,



if f and g are étale then so is h, while if h and f are étale then so is g. Thus $Sh(X)$ inherits all the relevant exactness properties from topological spaces. For the set of generators, one can take the collection of all embeddings $U \hookrightarrow X$ of open subsets of X. To see that these generate, take two distinct parallel maps a and b between sheaves



Let $e \in E$ be a point with $a(e) \neq b(e)$, and let V_e be a small neighbourhood of e so that $f : V_e \to f(V_e) = U$ is a homeomorphism onto the open set U. Then f^{-1} defines a map of sheaves from $(U \hookrightarrow X)$ to $(E \to X)$ with the property that $a \circ f^{-1} \neq b \circ f^{-1}$.

From the topos $Sh(X)$ of sheaves on X, one can recover the lattice $\mathcal{O}(X)$ of open subsets of X, essentially as the subcategory consisting of all sheaves $(E \to X)$ with the property that the unique map into the terminal object $1 = (id : X \to X)$ of $Sh(X)$ is a monomorphism. Thus we can recover the space X from $Sh(X)$ provided the points of X are determined by their open neighbourhoods. This is the case precisely when the space X is *sober*. [Recall, from SGA IV, vol. 1, p. 336, that a closed set F in X is irreducible if it cannot be written as the union of two smaller closed sets, and that X is sober if *every* such irreducible closed set F is of the from $F = \overline{\{x\}}$ for a unique point x. Every Hausdorff space is sober.] *In this text, all spaces will be assumed to be sober.*

A continuous map $f : Y \to X$ between spaces induces two well-known adjoint functors

$$f^* : Sh(X) \to Sh(Y) \quad , \quad f_* : Sh(Y) \to Sh(X)$$

between the categories of sheaves of sets. In terms of étale spaces, f^* is simply pullback (fibered product) along f. It evidently preserves (finite) limits and colimits. The right adjoint f_* is more easily described in terms of sheaves as functors: for a sheaf $F : \mathcal{O}(Y)^{op} \to (sets)$,

$$f_*(F) = F \circ f^{-1} : \mathcal{O}(X)^{op} \to \mathcal{O}(Y)^{op} \to (sets) .$$

These two functors constitute a morphism of topoi, denoted

$$f : Sh(Y) \to Sh(X) .$$

Conversely, suppose $\varphi : Sh(Y) \to Sh(X)$ is any morphism of topoi. Then the functor φ^*, when restricted to subobjects of the terminal object, gives an operation $\varphi^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ which preserves finite intersections and arbitrary unions. For a point $y \in Y$, define $F_y = X - \bigcup\{U \in \mathcal{O}(X) : y \notin \varphi^*(U)\}$. Then F_y is an irreducible closed set, so if Y is sober there is a unique point $x = \varphi(y)$ so that $F_y = \overline{\{x\}}$. This defines a map $\varphi : Y \to X$ with the property that for any open set $U \subseteq X$, and any point $y \in Y$, $\varphi(y) \in U$ iff $y \in \varphi^*(U)$. In this way, the map $\varphi : Y \to X$ is determined by the inverse image functor φ^*.

For sober spaces X and Y, these constructions set up a correspondence between continuous maps $Y \to X$ and (isomorphism classes of) topos morphisms $Sh(Y) \to Sh(X)$. Thus, the assignment

$$X \mapsto Sh(X)$$

of the topos of sheaves to a sober space X doesn't change the notion of mapping, and the topos $Sh(X)$ should simply be viewed as a faithful image of the space X in the world of topoi. Indeed, we will in the sequel often simply write X when it is evident that we mean the topos of sheaves on the space X. For example, when $\mathcal{E}$ is another topos, an arrow

$$X \to \mathcal{E}$$

denotes a topos morphism $Sh(X) \to \mathcal{E}$. In Section 4 below, we will discuss how algebraic invariants of the space X such as homotopy and cohomology groups can be defined in terms of the topos $Sh(X)$.

For the second elementary example of a topos, consider a small category $\mathbf{C}$. A *presheaf* (of sets) on $\mathbf{C}$ is a functor

$$S : \mathbf{C}^{op} \to (sets) .$$

Thus S assigns to each object $x \in \mathbf{C}$ a set $S(x)$, and to each arrow $\alpha : x \to y$ in $\mathbf{C}$ a function $S(\alpha) : S(y) \to S(x)$, called restriction along α and denoted

$$s \mapsto s \cdot \alpha = S(\alpha)(s) \quad (\text{for } s \in S(y)).$$

The functoriality of S is then reflected in the usual identities $s \cdot 1 = s$ and $(s \cdot \alpha) \cdot \beta = s \cdot (\alpha\beta)$ for an action. As morphisms $\varphi : S \to T$ between two such presheaves we take the natural transformations. Thus φ is given by functions $\varphi_x : S(x) \to T(x)$ (for each object x in $\mathbf{C}$), which respect the restrictions:

$$\varphi_x(s \cdot \alpha) = \varphi_y(s) \cdot \alpha ,$$

for s and α as above. This category of all presheaves on $\mathbf{C}$ is denoted as a functor category $sets^{\mathbf{C}^{op}}$, or as

$$\mathcal{B}\mathbf{C} .$$

This category $\mathcal{B}\mathbf{C}$ is a topos, called the *classifying topos* of the category $\mathbf{C}$. To see that the Giraud axioms are satisfied, note first that all limits and colimits of presheaves can be constructed "pointwise", as in

$$(\varprojlim S_i)(x) \;=\; \varprojlim S_i(x) \;\;,\;\; (\varinjlim S_i)(x) \;=\; \varinjlim S_i(x) \;.$$

Therefore all limits and colimits of presheaves, in particular pullbacks, sums and coequalizers, inherit all exactness properties from the category of sets. Thus it is clear that $\mathcal{B}\mathbf{C}$ satisfies the Giraud axioms (G1)-(G3).

For the axiom (G4) on generators, consider the "Yoneda embedding" $\mathrm{Yon} : \mathbf{C} \hookrightarrow \mathcal{B}\mathbf{C}$, defined by

$$\mathrm{Yon}(x)(y) \;=\; \mathrm{Hom}_{\mathbf{C}}(y, x) \;.$$

Thus $\mathrm{Yon}(x)$ is the *representable presheaf* given by x. The so-called Yoneda lemma states that for any presheaf S, there is a natural isomorphism

$$\theta = \theta_S : \mathrm{Hom}_{\mathcal{B}\mathbf{C}}(\mathrm{Yon}(x), S) \;\cong\; S(x) \;, \tag{1}$$

defined for a natural transformation $\varphi : \mathrm{Yon}(x) \to S$ by

$$\theta(\varphi) = \varphi_x\,(id_x) \;.$$

Naturality of θ means that for any morphism $\psi : S \to T$ of presheaves, the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}\mathbf{C}}(\mathrm{Yon}(x), S) & \xrightarrow{\;\theta_S\;} & S(x) \\
{\scriptstyle \psi_*}\downarrow & & \downarrow{\scriptstyle \psi_x} \\
\mathrm{Hom}_{\mathcal{B}\mathbf{C}}(\mathrm{Yon}(x), T) & \xrightarrow{\;\theta_T\;} & T(x)
\end{array}$$

commutes, where ψ_* denotes "composition with ψ". In particular, ψ is completely determined by all composites $\mathrm{Yon}(x) \xrightarrow{\varphi} S \xrightarrow{\psi} T$ from representable presheaves $\mathrm{Yon}(x)$. Thus, the collection of these presheaves, for all $x \in \mathbf{C}$, generates $\mathcal{B}\mathbf{C}$. This proves that $\mathcal{B}\mathbf{C}$ satisfies axiom (G4).

A functor $f : \mathbf{D} \to \mathbf{C}$ between small categories induces an evident operation f^* on presheaves, by composition:

$$f^* : \mathcal{B}\mathbf{C} \;\to\; \mathcal{B}\mathbf{D} \;\;,\;\; f^*(S)(y) = S(fy) \;.$$

This functor f^* evidently preserves limits and colimits, since these are all computed pointwise. Furthermore, f^* has a right adjoint $f_* : \mathcal{B}\mathbf{D} \to \mathcal{B}\mathbf{C}$, defined by

$$f_*(T)(x) = \mathrm{Hom}_{\mathcal{B}\mathbf{D}}(f^*(\mathrm{Yon}(x)), T) \, .$$

The adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{B}\mathbf{D}}(f^*(S), T) \;\cong\; \mathrm{Hom}_{\mathcal{B}\mathbf{C}}(S, f_*(T))$$

can be described as follows: given $\varphi : f^*(S) \to T$, construct $\bar{\varphi} : S \to f_*(T)$ with components $\bar{\varphi}_x : S(x) \to f_*(T)(x)$ defined via the isomorphism θ in (1) as

$$\begin{array}{ccccc}
S(x) & & \overset{\bar{\varphi}_x}{\dashrightarrow} & & f_*(T)(x) \\
\downarrow{\scriptstyle \theta^{-1}} & & & & \| \\
\mathrm{Hom}_{\mathcal{B}\mathbf{C}}(\mathrm{Yon}(x), S) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{B}\mathbf{D}}(f^*\mathrm{Yon}(x), f^*S) & \xrightarrow{\varphi_*} & \mathrm{Hom}_{\mathcal{B}\mathbf{D}}(f^*\mathrm{Yon}(x), T)
\end{array}$$

where φ_* denotes composition with φ. Conversely, given $\psi : S \to f_*(T)$, construct $\tilde{\psi} : f^*(S) \to T$ with components $\tilde{\psi}_y : f^*(S)(y) = S(fy) \to T(y)$, using the evident map $\mathrm{Yon}(y) \to f^*(\mathrm{Yon}(fy))$, as

$$\begin{array}{ccccc}
S(fy) & & \overset{\tilde{\psi}_y}{\dashrightarrow} & & T(y) \\
\downarrow{\scriptstyle \psi_{fy}} & & & & \uparrow{\scriptstyle \theta^{-1}} \\
f_*(T)(fy) & = & \mathrm{Hom}(f^*(\mathrm{Yon}(fy)), T) & \longrightarrow & \mathrm{Hom}(\mathrm{Yon}(y), T).
\end{array}$$

Thus the functor $f : \mathbf{D} \to \mathbf{C}$ induces a morphism of topoi, (again) denoted

$$f : \mathcal{B}\mathbf{D} \to \mathcal{B}\mathbf{C} \, ,$$

given by these adjoint functors f^* and f_*.

This construction of a topos morphism $\mathcal{B}\mathbf{D} \to \mathcal{B}\mathbf{C}$ from a functor $\mathbf{D} \to \mathbf{C}$ extends to natural transformations. Indeed, a transformation $\tau : g \to f$ between two functors $f, g : \mathbf{D} \rightrightarrows \mathbf{C}$ induces another transformation

$$\tilde{\tau} : f^* \to g^* : \mathcal{B}\mathbf{C} \rightrightarrows \mathcal{B}\mathbf{D} \, ,$$

defined for a presheaf S on $\mathbf{C}$ and an object y in $\mathbf{D}$ by

$$(\tilde{\tau}_S)_y : f^*(S)(y) = S(fy) \overset{S(\tau_y)}{\to} S(gy) = g^*(S)(y) \, .$$

Unlike the case of (sober) topological spaces, it is not true that all topos morphisms $\mathcal{B}\mathbf{D} \to \mathcal{B}\mathbf{C}$ come from functors $\mathbf{D} \to \mathbf{C}$. Indeed, there are many more morphisms $\mathcal{B}\mathbf{D} \to \mathcal{B}\mathbf{C}$ then there are functors $\mathbf{D} \to \mathbf{C}$, as will be evident from Chapter II, Section 2. In general, one cannot reconstruct the category $\mathbf{C}$ from the presheaf topos $\mathcal{B}\mathbf{C}$ either, because the representable presheaves are not characterized by a purely categorical property. (The closest one gets is by considering the class of all projective and connected presheaves: These are exactly the retracts of representable presheaves. If all idempotents split in $\mathbf{C}$, then every such retract is itself representable, up to isomorphism.)