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Lecture Notes in Mathematics

Stochastic Analysis and Related Topics II

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H. Korezlioglu A.S. Ustunel (Eds.)

Stochastic Analysis and Related Topics II

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FOREWORD

This volume contains the contributions of the participants to the second meeting on Stochastic Analysis and Related Topics, held in Silivri from July 18 to July 30, 1988, at the Nazim Terzioglu Graduate Research Center of University of Istanbul.

The first week of the meeting was devoted to the following lectures :

- Short Time Asymptotic Problems in Wiener Functional Integration Theory. Applications to Heat Kernels and Index Theorems, by S. Watanabe (Kyoto, Japan).

- Applications of Anticipating Stochastic Calculus to Stochastic Differential Equations, by E. Pardoux (Marseille, France).

- Wave Propagation in Random Media, by G. Papanicolaou (Courant Institute, New York, USA).

The lecture notes are presented at the beginning of the volume. We regret the absence of the lecture notes by G. Papanicolaou, who was too overloaded at the time. The presentation of the papers contributed to the volume ranges from the construction of new distribution spaces on the Wiener space to large deviations and random fields.

We would herewith like to thank the Scientific Direction of the ENST for its support in the preparation of the meeting and the present volume.

During the year of this meeting we lost our dear friend and colleague Michel METIVIER ; we are dedicating this volume to his memory.

H. KOREZLIOGLU A.S. USTUNEL

CONTENTS

SHORT TIME ASYMPTOTIC PROBLEMS IN WIENER FUNCTIONAL INTEGRATION THEORY. APPLICATIONS TO HEAT KERNELS AND INDEX THEOREMS.

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INTRODUCTION.

Since the Wiener measure space was introduced by N. Wiener in 1923, a rigorous theory of path space integrals has been developed with many interesting applications to mathematics and mathematical physics. Especially, the Feynman-Kac formula was established by M. Kac and it was applied to several problems in the spectral theory of Schrodinger operators and potential theory. If we want to extend Kac's theory to curved Riemannian spaces, we need to make use of an important stochastic calculus on the Wiener space, that is, Ito's stochastic calculus. Indeed, such important notions as Brownian motions and stochastic moving frames on Riemannian manifolds can be constructed by solving Ito's stochastic differential equations.

The main purpose of my lecture is to discuss this probabilistic approach by the Wiener functional integration to obtain short time asymptotics of traces (supertraces) of heat kernels. It is well-known that many important problems in analysis, geometry and mathematical physics, such as asymptotics of eigenvalues of the Laplacian, index theorems, fixed point formulas, Morse inequalities for Morse functions, Poincare-Hopf index theorem for vector fields and so on are essentially related to this problem of estimating traces of heat kernels. The method of Wiener functional integration consists of first representing the heat kernels by integrals of certain Wiener functionals and then study the asymptotics of functionals by probabilistic techniques. There is however a crucial difficulty in this approach. Heat kernels, i.e. fundamental solutions of heat equations, can not be represented by an ordinary expectation of Wiener functionals but by a conditional expectation, which, *in a standard theory of probability, is defined in almost everywhere sense.* Thus an disintegration theory, i.e. a refinement of conditional expectations, is needed in this approach. One approach given so far is to use *pinned diffusion*, or *tied-down diffusion processes*. But the very definition of tied-down diffusion involves fundamental solution of heat equation in essential way so a kind of tautology occurs and some analytical knowledge of heat kernels is inevitable in this approach. Here we appeal to the *Malliavin calculus* for this disintegration problem. Namely, we introduce a family of Sobolev spacec of Wiener functionals by refining the usual family of L_p-spaces. Among these Wiener functinals are *generalized Wiener functionals*, an analogue of Schwartz distributions over the Wiener space. Similarly as in the Schwartz distribution theory, we can generalize the notion of expectations to these generalized Wiener functional. Using these notions , the disintegration problem for Wiener functional integration can be well discussed. Very roughly, our approach systematically used in this lecture may be described as follows. We represent the quantity $p(\varepsilon)$ (typically a heat kernel or the trace of a heat kernel) for which we would estimate the asymptotic with respect to the parameter ε as $p(\varepsilon)=E(\Phi(\varepsilon))$ by a generalized Wiener functional expectation of a generalized Wiener functional $\Phi(\varepsilon)$. We decompose $\Phi(\varepsilon)$ as $\Phi(\varepsilon) = \Phi_1(\varepsilon) + \Phi_2(\varepsilon)$ in such a way that for $\Phi_2(\varepsilon)$, we can estimate its Sobolev norm and thereby show that it is negligible, secondly $\Phi_1(\varepsilon)$ is a generalized Wiener functional having a rather simple structure so that we can manage to compute the generalized expectation $E(\Phi_1(\varepsilon))$ explicitly.

Finally, we explain the content of this lecture. In §1, we review the fundamental concepts and results in the Malliavin calculus; the Sobolev spaces of Wiener functionals and differential calculus defined on them, pull-back of Schwartz distributions by finite dimensional non-degenerate Wiener maps, dependence on parameters, especially, the asymptotic expansion of Wiener functionals and so on. In §2, we discuss the application of the Malliavin calculus to Ito functionals, an important case of Wiener functionals defined by solutions of stochastic differential equations. To illustrate our method, we reproduce in §3 some results of McKean-Singer [30] for heat kernels on a compact Riemannian manifold with and without boundary. In §4, we give a proof of index theorems by our probabilistic method.

§1. A survey of the Malliavin calculus

1.1 *Sobolev spaces of Wiener functionals and generalized Wiener functionals.*

Let (W_0^r, P) be the r-dimensional Wiener space:

$$W_0^r = \{ w \in C([0,1] \to R^r);\ w(0)=0 \}$$

endowed with the supremum norm and P is the standard Wiener measure on W_0^r. W_0^r is denoted simply by W in the sequel. We restrict ourselves to the Wiener space with the time interval $[0,1]$ just because of simplicity and that it is sufficient in the problems discussed here.

By a Wiener functional, we mean a P-measurable function on W, more precisely, an equivalence class of P-measurable functions coinciding with each other P-almost surely. Let $L_p, 1 \le p < \infty$, be the usual L^p-space of real-valued Wiener functionals with the L_p-norm

$\| \ \|_p$. If we set

(1.1) $\quad L_{\infty-} = \cap_{1<p<\infty} L_p,$

then $L_{\infty-}$ is a Frechet space and it is an algebra, i.e. if $f, g \in L_{\infty-}$, then $f \cdot g \in L_{\infty-}$. Its dual is clearly $L_{1+} = \cup_{1<p<\infty} L_p$. More generally, if E is a real separable Hilbert space, we denote by $L_p(E)$ the real L_p-space of E-valued Wiener functionals and by $\| \ \|_{p,E}$, or simply by $\| \ \|_p$ when there is no confusion, the norm of $L_p(E)$. Thus $L_p(R) = L_p$. $L_{\infty-}(E)$ and $L_{1+}(E)$ are defined similarly.

Let H be the Cameron-Martin Hilbert subspace of W formed of all $w \in W$ which are absolutely continuous in $t \in [0,1]$ with square integrable derivatives and endowed with the norm

(1.2) $\quad \|w\|_H^2 = \int_0^1 |\frac{dw}{dt}|(t)^2 dt$

We identify the dual H' of H with H by the Riesz theorem and then $W' \subsetneq H' = H \subsetneq W$ where $\subsetneq$ denotes the continuous inclusion. For $h \in H$, define $[h](w) \in L_2$ by the usual Wiener integral

(1.3) $\quad [h](w) = \int_0^1 \frac{dh}{dt}(s) \cdot dw(s) = \sum_{\alpha=1}^{r} \int_0^1 \frac{dh^\alpha}{dt}(s) dw^\alpha(s)$.

Then $([h](w);\ h \in H)$ $(\subset L_2)$ is a mean zero Gaussian system with the covariance $E([h](w) \cdot [h'](w)) = \langle h, h' \rangle_H$. A Wiener functional $F(w)$ is called a polynomial functional if there exist a real polynomial $p(t_1, \ldots, t_n)$ and $h_1, \ldots, h_n \in H$ such that

$$F(w) = p([h_1](w), \ldots, [h_n](w)).$$

The totality of polynomial functionals is denoted by P. Clearly $P \subset L_{\infty-}$ and this inclusion is dense. More generally, given a separable Hilbert space E, an E-valued polynomial functional $F(w)$ is any E-valued Wiener functional expressible in the form of a finite sum $F(w) = \sum F_i(w) e_i$, $F_i \in P$, $e_i \in E$. The totality of E-valued polynomial functionals is denoted by $P(E)$. Then

$P(E) \subset L_{\infty-}(E)$ and the inclusion is dense. Now $L_2(E)$ is a Hilbert space and is decomposed into a direct sum of mutually orthogonal subspaces of Wiener's homogeneous chaos:

$$(1.4) \qquad L_2(E) = C_0(E) \oplus C_1(E) \oplus \ldots \oplus C_n(E) \oplus \ldots$$

The projection of $L_2(E)$ onto $C_n(E)$ is denoted by J_n. If $F \in P(E)$ then $J_n F \in P(E)$ and $F = \sum J_n F$ is actually a finite sum. Therefore, operators L and $(I-L)^s$, $s \in R$, from $P(E)$ into itself can be defined by

$$LF = \sum_n (-n) J_n F \qquad \text{and} \qquad (I-L)^s F = \sum_n (1+n)^s J_n F.$$

For $1<p<\infty$ and $s \in R$, define a norm $\| \ \|_{p,s}$ (denoted also by $\| \ \|_{p,s;E}$ to make the value space clear) by

$$(1.5) \qquad \|F\|_{p,s} = \|(I-L)^{s/2} F\|_p \qquad (\| \ \|_p \text{ is the } L_p\text{-norm of } L_p(E))$$

These norms have the following basic properties

(i)(monotonicity)

$$\|F\|_{p,s} \le \|F\|_{p',s'} \quad \text{for any } F \in P(E) \text{ if } p \le p' \text{ and } s \le s'$$

(ii)(compatibility) If $(F_n) \subset P(E)$ satisfies $\|F_n - F_m\|_{p,s} \to 0$ as $n,m \to \infty$ and $\|F_n\|_{p',s'} \to 0$ as $n \to \infty$ then $\|F_n\|_{p,s} \to 0$ as $n \to \infty$.

(iii)(duality) For $G \in P(E)$ and $1<p<\infty$, $s \in R$,

$$\|G\|_{p,s} = \sup\left(\int_W \langle G(w), F(w) \rangle_E \, P(dw) \ ; \ F \in P(E), \ \|F\|_{q,-s} \le 1 \right)$$

where $p^{-1} + q^{-1} = 1$.

Let $D_p^s(E)$ be the completion of $P(E)$ with respect to $\| \ \|_{p,s}$. Then the above properties of the norm imply the following:

$$(1.6) \qquad D_p^0(E) = L_p(E)$$

$$(1.7) \qquad D_{p'}^{s'}(E) \subsetneq D_p^s(E) \quad \text{if } p \le p' \text{ and } s \le s'$$

$$(1.8)\qquad D_p^s(E)' = D_q^{-s}(E), \qquad p^{-1}+q^{-1}=1.$$

Thus an element in $D_p^s(E)$ for $s\geq 0$, being an element in $L_p(E)$, is a Wiener functional in the usual sense, but some elements in $D_p^s(E)$ for $s<0$ are no longer so but a kind of Schwartz distribution on the Wiener space. We call such elements as generalized Wiener functionals. We set $D^\infty(E) = \cap_{s>0} \cap_{1<p<\infty} D_p^s(E)$ and $D^{-\infty}(E) = \cup_{s>0} \cup_{1<p<\infty} D_p^{-s}(E)$. Again we denote them simply by D^∞ and $D^{-\infty}$ when $E = R$.

Let E_1 and E_2 be real separable Hilbert spaces. For any $p,q \in (1,\infty)$ and $k = 0,1,2,\ldots$ such that $p^{-1}+q^{-1} = r^{-1}< 1$, there exists a positive constant $C_{p,q,k}$ such that

$$(1.9)\qquad \|F\otimes G\|_{r,k;E_1\otimes E_2} \leq C_{p,q,k} \|F\|_{p,k;E_1}\|G\|_{q,k;E_2}$$

for every $F \in D_p^k(E_1)$, $G \in D_q^k(E_2)$

From this it follows that D^∞ is an algebra i.e. if $F, G \in D^\infty$ then $F\cdot G \in D^\infty$. More generally, if $F \in D^\infty$ and $G \in D^\infty(E)$, then $F\cdot G \in D^\infty(E)$. Thus $D^\infty(E)$ is a D^∞-module. Also, if $F \in D^\infty$ and $\Phi \in D^{-\infty}(E)$, then $F\cdot\Phi \in D^{-\infty}(E)$ is defined by

$$(1.10)\qquad \langle F\cdot\Phi, G\rangle = \langle \Phi, F\cdot G\rangle \qquad \text{for every } G \in D^\infty(E),$$

where $\langle\ ,\ \rangle$ is the canonical bilinear form on $D^{-\infty}(E) \times D^\infty(E)$. Thus $D^{-\infty}(E)$ is also a D^∞-module. The continuity of this multiplication is more precisely stated in the following inequality which immediately follows from (1.9): For every $p,q \in (1,\infty)$ such that $p^{-1}+q^{-1}=r^{-1}<1$ and $k = 0,1,2,\ldots$, there exists a positive constant $C'_{p,q,k}$ such that

$$(1.11)\qquad \|F\cdot\Phi\|_{r,-k;E} \leq C'_{p,q,k}\|F\|_{p,k}\|\Phi\|_{q,-k;E}$$

for every $F \in D_p^k$ and $\Phi \in D_p^{-k}(E)$.

If 1 is the Wiener functional taking the constant value 1, then clearly $1 \in D^{\infty}$. For $\Phi \in D^{-\infty}$, the natural coupling $\langle \Phi, 1 \rangle$ is denoted by $E(\Phi)$ and called the generalized expectation of Φ. If $\Phi \in L_{1+} \subset D^{-\infty}$, $E(\Phi)$ coincides with the ordinary expectation $\int_W \Phi(w) P(w)$. Note that for every $F \in D^{\infty}$ and $\Phi \in D^{-\infty}$, the generalized expectation $E(F \cdot \Phi)$ coincides with the natural coupling $\langle F, \Phi \rangle$. It is clear from the definition that

(1.12) $$|E(\Phi)| \le \text{const.} \|\Phi\|_{p,-k} \quad \text{if} \quad \Phi \in D_p^{-k},\ 1<p<\infty,\ k = 1,2,...$$

For $F \in P(E)$, its Frechet derivative $DF(w) \in P(H \otimes E)$ is defined by

(1.13) $$DF(w)(h) = \lim_{\varepsilon \downarrow 0} \frac{F(w+\varepsilon h)-F(w)}{\varepsilon}.$$

(Note that $H \otimes E$ is the Hilbert space formed of all linear opertors $H \to E$ of the Hilbert-Schmidt type endowed with the Hilbert-Schmidt norm). If $G \in P(H \otimes E)$ is such that it is expressed in the form of a finite sum

(1.14) $$G(w) = \sum_{i,j} G_{ij}(w) \cdot h_i \otimes e_j, \qquad G_{ij} \in P, \quad h_i \in H, \quad e_j \in E$$

where $h_i \otimes e_j \in H \otimes E$ is defined by

$$(h_i \otimes e_j)(h) = \langle h, h_i \rangle_H \cdot e_j ,$$

we define $[G] \in P(E)$ by

(1.15) $$[G](w) = \sum_{i,j} G_{ij}(w) \cdot [h_i](w) \cdot e_j$$

In the expression (1.14), we may assume without loss of generality that $(h_i) \subset H$ are orthnormal and then trace $DG \in P(E)$ is given by

(1.16) $$\text{trace } DG(w) = \sum_{i,j} DG_{ij}(w)(h_i) \cdot e_j$$

We define, for $G \in P(H \otimes E)$ expressible in the form (1.14), $D^{*}G \in P(E)$ by

(1.17) $$D^{*}G(w) = -\text{trace } DG(w) + [G](w)$$

By an integration by parts in a finite dimensional Gaussian measure integration, we can prove that

(1.18) $$\int_W \langle DF(w),\ G(w)\rangle_{H\otimes E}\ P(dw) = \int_W \langle F(w),\ D^*G(w)\rangle_E\ P(dw)$$

for $F \in P(E)$ and $G \in P(H\otimes E)$ of the form (1.14).

Also, for $F \in P(E)$, $DF \in P(H\otimes E)$ is always expressible in the form (1.14) and it holds that

(1.19) $$LF(w) = -D^*(DF)(w)$$

It is an important result of Meyer that the operator

$$D : P(E) \to P(H\otimes E)$$

is uniquely extended to a linear operator

$$D : D^{-\infty}(E) \to D^{-\infty}(H\otimes E)$$

which is continuous in the sense that its restriction $D_p^{s+1}(E) \to D_p^s(H\otimes E)$ is continuous for every $p \in (1,\infty)$ and $s \in R$. Consequently by taking the dual, the operator D^* defined by (1.17) on G of the form (1.14) is uniquely extended to a linear operator $D^{-\infty}(H\otimes E) \to D^{-\infty}(E)$ which is continuous in the sense that its restriction $D_p^{s+1}(H\otimes E) \to D_p^s(E)$ is continuous for every $p \in (1,\infty)$ and $s \in R$. Actually D^* is the dual of D. By the definition of the Sobolev spaces, it is clear that L is uniquely extended to a linear operator $D^{-\infty}(E) \to D^{-\infty}(E)$ such that its restriction $D_p^{s+2}(E) \to D_p^s(E)$ is continuous for every $p \in (1,\infty)$ and $s \in R$. Furthermore, (1.19) holds. Also (1.18) can be extended obviously in the context of Sobolev spaces. Finally we remark another important result of Meyer that the norm $\|F\|_{p,k;E}$ for $1<p<\infty$ and $k = 0,1,2,\dots,$ is equivalent to the norm $\sum_{i=0}^k \|D^iF\|_{p;H\otimes \overset{i}{H\otimes}\dots\otimes H\otimes E}$. For details of the above facts and the chain rules of D, D^* and L, cf. [31],[37],[40].

1.2 *Pull-back of Schwartz distributions and disintegration of Wiener functional integration*

Let $F : W \to R^d$ be a d-dimensional Wiener functional. It is said to be smooth in the sense of Malliavin if $F \in D^\infty(R^d)$, i.e. $F = (F^1,\dots,F^d)$ with $F^i \in D^\infty$. In this case

(1.20) $\sigma^{ij}(w) = \langle DF^i(w), DF^j(w)\rangle_H \in D^\infty, \quad i,j = 1,\dots,d$

The Wiener functional $\sigma = \sigma_F = (\sigma^{ij})$ with values in nonnegative definite symmetric d×d-matrices is called the Malliavin covariance of F. F is said to be nondegenerate in the sense of Malliavin if

(1.21) $[\det \sigma(w)]^{-1} \in L_{\infty-} = \cap_{1<p<\infty} L_p$

In this case $\gamma = (\gamma^{ij}) = \sigma^{-1}$ satisfies $\gamma^{ij} \in D^\infty$.

Suppose that we are given $F \in D^\infty(R^d)$ satisfying the nondegeneracy condition (1.21). We show that every Schwartz distribution $T(x)$ on R^d can be lifted or pulled-back to a generalized Wiener functional $T\circ F$ (denoted also by $T(F)$) in $D^{-\infty}$ under the Wiener map $F : W \to R^d$. For this we introduce the following family of real Banach spaces of functions and generalized functions on R^d. Let $\mathcal{S}(R^d)$ be the real Schwartz space of rapidly decreasing C^∞-functions on R^d and set

$$\|\varphi\|_{2k} = \|(1+|x|^2-\Delta)^k\varphi\|_\infty, \quad \varphi \in \mathcal{S}(R^d), \quad k = 0,\pm1,\pm2,\dots,$$

where $\|\ \|_\infty$ is the supremum norm and $\Delta = \Sigma_{i=1}^d (\partial/\partial x^i)^2$. Let $\mathcal{T}_{2k}$ be the completion of $\mathcal{S}(R^d)$ by the norm $\|\ \|_{2k}$. Then we have

$$\mathcal{S}(R^d) \subsetneq \dots \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_0 \subsetneq \mathcal{T}_{-2} \subsetneq \dots \subsetneq \mathcal{S}'(R^d)$$

and $\mathcal{T}_0 = \hat{C}(R^d)$:= the Banach space of all real continuous functions on R^d tending to 0 at infinity endowed with the supremum norm. Furthermore $\cap_{k=1}^\infty \mathcal{T}_{2k} = \mathcal{S}(R^d)$ and $\cup_{k=1}^\infty \mathcal{T}_{-2k} = \mathcal{S}'(R^d)$.

THEOREM 1.1 ([15],[40]) Let $F \in D^{\infty}(R^d)$ be given and satisfy the non-degeneracy condition (1.21). Then for every $p \in (1,\infty)$ and $k = 0,1,2,\ldots,$ there exists a positive constant $C = C_{p,k,F}$ such that

$$(1.22) \qquad \|\varphi \circ F\|_{p,-2k} \leq C \|\varphi\|_{-2k} \quad \text{for all} \quad \varphi \in \mathcal{S}(R^d)$$

(Note that $\varphi \circ F \in D^{\infty}$.)

Hence the map $\varphi \in \mathcal{S}(R^d) \to \varphi \circ F \in D^{\infty}$ can be extended uniquely to a linear map

$$T \in \mathcal{S}'(R^d) \to T \circ F \in D^{-\infty}$$

such that its restriction $T \in \mathcal{T}_{-2k} \to T \circ F \in D_p^{-2k}$ is continuous for every $p \in (1,\infty)$ and $k = 0,1,2,\ldots$ In particular, $T \circ F \in \tilde{D}^{-\infty} := \cup_{k=1}^{\infty} \cap_{1<p<\infty} D_p^{-k}$ for every $T \in \mathcal{S}'(R^d)$.

$T \circ F$, denoted also by $T(F)$, is called the <u>composition of a Schwartz distribution</u> $T \in \mathcal{S}'(R^d)$ <u>and</u> F, or the <u>pull-back of</u> T <u>under the Wiener map</u> $W \to R^d$. Note that $\tilde{D}^{-\infty}$ is much smaller than $D^{-\infty}$ and for any G in $\tilde{D}^{\infty} := \cap_{k=1}^{\infty} \cup_{1<p<\infty} D_p^{k}$, which is much larger than D^{∞}, $G \cdot T \circ F \in D^{-\infty}$ is well-defined and hence the generalized expectation $E[G \cdot T \circ F]$ is well defined.

Using this notion of the pull-back, the disintegration problem can be discussed as follows. Suppose that $F: W \to R^d$ satisfy the same assumptions as in Th.1.1. Noting that δ_x, the Dirac δ-function at $x \in R^d$, is in $\mathcal{T}_{-2m}$ for $m \geq m_0$, where $m_0 = [d/2]+1$, we see from Th.1.1 that, for $k = 0,1,2,\ldots,$

$$x \in R^d \to \delta_x(F) \in D_p^{-2m_0-2k}$$

is continuously differentiable 2k-times. Hence, for every $G \in \cup_{1<p<\infty} D_p^{2m_0+2k}$,

$$x \in R^d \to E[G \cdot \delta_x(F)]$$

is C^{2k} and therefore it is C^∞ if $G \in \tilde{D}^\infty$. In particular, $p_F(x) = E[\delta_x(F)]$ is a C^∞-function on R^d. But this $p_F(x)$ is the density , with respect to the Lebesgue measure on R^d, of the law of the d-dimensional Wiener functional F, as is seen from

$$\int_{R^d} p_F(x)\varphi(x)dx = \int_{R^d} \varphi(x)E[\delta_x(F)]dx = E[\int_{R^d}\varphi(x)\delta_x(F)dx]$$

$$= E[\varphi(F)].$$

In this way we have deduced that the law of F has a C^∞-density if $F \in D^\infty(R^d)$ satisfies the nondegeneracy condition (1.21). Furthermore, it is easy to see that $E[G\cdot\delta_x(F)]$ is a version of $E[G|F=x]p_F(x)$. Thus, on a set where p_F is positive, the conditional expectation of $G \in \tilde{D}^\infty$ given F has a smooth version.

1.3 *Asymptotic evaluation of (generalized) Wiener functional expectations.*

Let $(\Phi(\varepsilon,w))$ be a family of generalized Wiener functionals $\Phi(\varepsilon,w) \in D^{-\infty}(E)$ depending on a parameter $\varepsilon \in (0,1]$. We can speak of its asymptotics as $\varepsilon\downarrow 0$ in terms of Sobolev spaces, e.g., we say that

$$\Phi(\varepsilon,w) = O(\varepsilon^k) \quad \text{or} \quad = o(\varepsilon^k) \quad \text{as} \quad \varepsilon\downarrow 0 \quad \text{in} \quad D_p^s(E)$$

accordingly as

$$\limsup_{\varepsilon\downarrow 0} \|\Phi(\varepsilon,w)\|_{p,s;E}/\varepsilon^k < \infty \quad \text{or} \quad = 0 \quad \text{as} \quad \varepsilon\downarrow 0$$

where k is some real constant and, here we adopt the convention that $\|\Phi(\varepsilon,w)\|_{p,s;E} = \infty$ if $\Phi(\varepsilon,w) \notin D_p^s(E)$. Based on this notion, it is natural to define that $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $D^\infty(E)$ as $\varepsilon\downarrow 0$ if $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $D_p^s(E)$ as $\varepsilon\downarrow 0$ for every $s>0$ and $p\in(1,\infty)$. Also we say that $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $\tilde{D}^\infty(E)$ as $\varepsilon\downarrow 0$ if for every $s>0$ there exists $p\in(1,\infty)$ such that $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $D_p^s(E)$ as $\varepsilon\downarrow 0$. Similarly we say that

$\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $\tilde{D}^{-\infty}(E)$ if $s>0$ exists such that $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $D_p^{-s}(E)$ for all $1<p<\infty$.

Finally we say that $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $D^{-\infty}(E)$ as $\varepsilon\downarrow 0$ if $\Phi(\varepsilon,w) = O(\varepsilon^k)$ in $D_p^{-s}(E)$ as $\varepsilon\downarrow 0$ for some $s>0$ and $p\in(1,\infty)$. It is easy to see from (1.9) and (1.11) that if $F(\varepsilon,w)= O(\varepsilon^k)$ in D^∞ and $G(\varepsilon,w)= O(\varepsilon^m)$ in $D^\infty(E)$ ($\tilde{D}^\infty(E)$) as $\varepsilon\downarrow 0$, then $F(\varepsilon,w)\cdot G(\varepsilon,w)= O(\varepsilon^{k+m})$ in $D^\infty(E)$ (resp. $\tilde{D}^\infty(E)$) as $\varepsilon\downarrow 0$. Also if $F(\varepsilon,w)= O(\varepsilon^k)$ in $\tilde{D}^\infty$ and $G(\varepsilon,w)= O(\varepsilon^m)$ in $D^\infty(E)$ as $\varepsilon\downarrow 0$ then $F(\varepsilon,w)\cdot G(\varepsilon,w)= O(\varepsilon^{k+m})$ in $\tilde{D}^\infty(E)$ as $\varepsilon\downarrow 0$. Furthermore if $F(\varepsilon,w)= O(\varepsilon^k)$ in D^∞ and $\Phi(\varepsilon,w) = O(\varepsilon^m)$ in $D^{-\infty}(E)$ ($\tilde{D}^\infty(E)$) as $\varepsilon\downarrow 0$ then $F(\varepsilon,w)\Phi(\varepsilon,w)= O(\varepsilon^{k+m})$ in $D^{-\infty}(E)$ (resp. $\tilde{D}^{-\infty}(E)$) as $\varepsilon\downarrow 0$ and if $F(\varepsilon,w)= O(\varepsilon^k)$ in $\tilde{D}^\infty$ and $\Phi(\varepsilon,w)= O(\varepsilon^m)$ in $\tilde{D}^{-\infty}(E)$ as $\varepsilon\downarrow 0$ then $F(\varepsilon,w)\cdot\Phi(\varepsilon,w) = O(\varepsilon^{k+m})$ in $D^{-\infty}(E)$ as $\varepsilon\downarrow 0$. It is easy to see that if

$$\Phi(\varepsilon,w) = O(\varepsilon^k) \quad \text{in } D^{-\infty}(E) \quad \text{as } \varepsilon\downarrow 0$$

then its generalized expectation satisfies

$$E(\Phi(\varepsilon,w)) = O(\varepsilon^k) \quad \text{as } \varepsilon\downarrow 0$$

in the ordinary numerical sense.

We say

$$F(\varepsilon,w)\sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \quad \text{in } D^\infty(E) \quad \text{as } \varepsilon\downarrow 0$$

if $f_0, f_1, f_2,\dots\in D^\infty(E)$ and for every n,

$$F(\varepsilon,w) - [f_0+\varepsilon f_1+\dots+\varepsilon^n f_n] = O(\varepsilon^{n+1}) \quad \text{in } D^\infty(E) \text{ as } \varepsilon\downarrow 0.$$

Similarly we can define

$$F(\varepsilon,w)\sim f_0 + \varepsilon f_1 + \dots \quad \text{in } \tilde{D}^\infty(E) \quad \text{as } \varepsilon\downarrow 0$$

and

$$\Phi(\varepsilon,w)\sim \phi_0 + \varepsilon\phi_1 + \dots \quad \text{in } \tilde{D}^{-\infty}(E) \text{ or in } D^{-\infty}(E) \quad \text{as } \varepsilon\downarrow 0.$$