Smooth Dynamical Systems

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Preface

In 1966 I began teaching a third year undergraduate course in the geometric theory of differential equations. This had previously been given by my friend and colleague Stewart Robertson (now of Southampton University). We both felt that no modern text book really covered the course, and we decided to collaborate in writing one. We had in mind something very simple, with plenty of pictures and examples and with clean proofs of some nice geometric results like the Poincaré–Bendixson theorem, the Poincaré–Hopf theorem and Liapunov's direct method. Unfortunately, over the years, this book stubbornly refused to materialize in a publishable form. I am afraid that I was mainly responsible for this. I became increasingly interested in detailed proofs and in presenting a coherent development of the basic theory, and, as a result, we lost momentum. Eventually two really excellent introductions (Arnold [1] and Hirsch and Smale [1]) appeared, and it is to these that one would now turn for an undergraduate course book. The point of this piece of history is to emphasize the very considerable contribution that Professor Robertson has made to the present book, for this has developed out of our original project. I am very happy to have the opportunity of thanking him both for this and also for his help and encouragement in my early years at Liverpool.

The book that has finally appeared is, I suppose, mainly for postgraduates, although, naturally, I should like to foist parts of it upon undergraduates as well. I hope that it will be useful in filling the gap that still exists between the above-mentioned text books and the research literature. In the first six chapters, I have given a rather doctrinaire introduction to the subject, influenced by the quest for generic behaviour that has dominated research in recent years. I have tried to give rigorous proofs and to sort out answers to questions that crop up naturally in the course of the development. On the other hand, in Chapter 7, which deals with some aspects of the rich flowering of the subject that has taken place in the last twenty-odd years, I have gone in for informal sketches of the proofs of selected theorems. Of course, the choice of results surveyed is very much a function of my own interests and, particularly, competence. This explains, for example, my failure to say anything much about ergodic theory or Hamiltonian systems.

I have tried to make the book reasonably self-contained. I have presupposed a grounding in several-variable differential calculus and a certain amount of elementary point set topology. Very occasionally results from algebraic topology are quoted, but they are of the sort that one happily takes on trust. Otherwise, the basic material (or, at least, enough of it to get by with) is contained in various slag heaps, labelled Appendix, that appear at the end of chapters and at the end of the book. For example, there is a long appendix on the theory of smooth manifolds, since one of the aims of the book is to help students to make the transition to the global theory on manifolds. The appendix establishes the point of view taken in the book and assembles all the relevant apparatus. Its later pages are an attempt to alleviate the condition of the student who shares my congenital inability to grasp the concept of affine connection. To make room for such luxuries, I have, with regret, omitted some attractive topics from the book. In particular, the large body of theory special to two dimensions is already well treated in text books, and I did not feel that I could contribute anything new. Similarly, there is not much emphasis on modelling applications of the theory, except in the introduction. I feel more guilty about ducking transversality theory, and this is, in part, due to a lack of steam. However, after a gestation period that would turn an Alpine black salamander green with envy, it must now be time to stand and deliver.

When working my way into the subject, I found that the books by Coddington and Levinson [1], Hurewicz [1], Lefschetz [1], Nemitskii and Stepanov [1] and, at a later stage, Abraham [1] and Abraham and Robbin [1] were especially helpful. I should like to express my gratitude to my colleague at Liverpool, Bill Newns, who at an early stage read several of the chapters with great care and insight. I am also indebted to Plinio Moreira, who found many errors in a more recent version of the text, and to Andy du Plessis for helpful comments on several points. Finally, a special thank-you to Jean Owen, who typed the whole manuscript beautifully and is still as friendly as ever.

CONTENTS

Introduction

In the late nineteenth century, Henri Poincaré created a new branch of mathematics by publishing his famous memoir (Poincaré [1]) on the qualitative theory of ordinary differential equations. Since then, differential topology, one of the principal modern developments of the differential calculus, has provided the proper setting for this theory. The subject has a strong appeal, for it is one of the main areas of cross-fertilization between pure mathematics and the applied sciences. Ordinary differential equations crop up in many different scientific contexts, and the qualitative theory often gives a major insight into the physical realities of the situation. In the opposite direction, substantial portions of many branches of pure mathematics can be traced back, directly or indirectly, to this source.

Suppose that we are studying a process that evolves with time, and that we wish to model it mathematically. The possible states of the system in which the process is taking place may often be represented by points of a differentiable manifold, which is known as the *state space* of the model. For example, if the system is a single particle constrained to move in a straight line, then we may take Euclidean space $\mathbf{R}^2$ as the state space. The point $(x, y) \in \mathbf{R}^2$ represents the state of the particle situated x units along the straight line from a given point in a given direction moving with a speed of y units in that direction. The state space of a model may be finite dimensional, as in the above case, or it may be infinite dimensional. For example, in fluid dynamics we have the velocity of the fluid at infinitely many different points to take into account and so the state space is infinite dimensional. It may happen that all past and future states of the system during the process are completely determined by its state at any one particular instant. In this case we say that the process is *deterministic*. The processes modelled in classical Newtonian mechanics are deterministic; those modelled in quantum mechanics are not.

In the deterministic context, it is often the case that the processes that can take place in the system are all governed by a smooth vector field on the state space. In classical mechanics, for example, the vector field involved is just another way of describing the *equations of motion* that govern all possible motions of the system. We can be more explicit as to what we mean by a vector field governing a process. As the process develops with time, the point representing the state of the system moves along a curve in the state space. The velocity of this moving point at any position x on the curve is a tangent vector to the state space based at x. The process is *governed* by the vector field if this tangent vector is the value of the vector field at x, for all x on the curve.

In the *qualitative* (or *geometric*) *theory*, we study smooth vector fields on differentiable manifolds, focusing our attention on the collection of parametrized curves on the manifold that have the tangency property described above. Our hope is that any outstanding geometrical feature of the curve system will correspond to a significant physical phenomenon when the vector field is part of a good mathematical model for a physical situation. This seems reasonable enough, and it is borne out in practice. We complete this motivational introduction by examining some familiar examples in elementary mechanics from this viewpoint. The remainder of the book is more concerned with the mathematical theory of the subject than with its modelling applications.

I. THE SIMPLE PENDULUM

Consider a particle P of mass m units fixed to one end of a rod of length l units and of negligible mass, the other end Q of the rod being fixed. The rod is free to rotate about Q without friction or air resistance in a given vertical plane through Q. The problem is to study the motion of P under gravity. The mechanical system that we have described is known as the *simple pendulum* and is already a mathematical idealization of a real life pendulum. For simplicity we may as well take $m = l = 1$, since we can always modify our units to produce this end. The first stage of our modelling procedure is completed by the assumption that gravity exerts a constant force on P of g units/sec^2 vertically downwards.

We now wish to find a state space for the simple pendulum. This is usually done by regarding the rotation of PQ about Q as being positive in one direction and negative in the other, and measuring

(i) the angular displacement θ radians of PQ from the downwards vertical through Q, and

(ii) the angular velocity ω radians/sec of PQ (see Figure 0.1).

We can then take $\mathbf{R}^2$ as the state space, with coordinates (θ, ω).

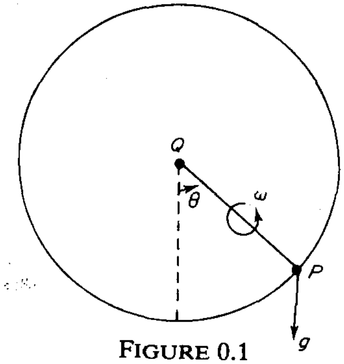

FIGURE 0.1

The equation of motion for the pendulum is

(0.2) $$\theta'' = -g \sin \theta,$$

where $\theta'' = d^2\theta/dt^2$. Using the definition of ω, we can replace this by the pair of first order equations

(0.3) $$\begin{aligned} \theta' &= \omega, \\ \omega' &= -g \sin \theta. \end{aligned}$$

A solution of (0.3) is a curve (called an *integral curve*) in the (θ, ω) plane parametrized by t. If the parametrized coordinates of the curve are $(\theta(t), \omega(t))$ then the tangent vector to the curve at time t is $(\omega(t), -g \sin \theta(t))$, based at the point $(\theta(t), \omega(t))$. We get various integral curves corresponding to various initial values of θ and ω at time $t = 0$, and these curves form the so-called *phase portrait* of the model. It can be shown that the phase portrait looks like Figure 0.4. One can easily distinguish five

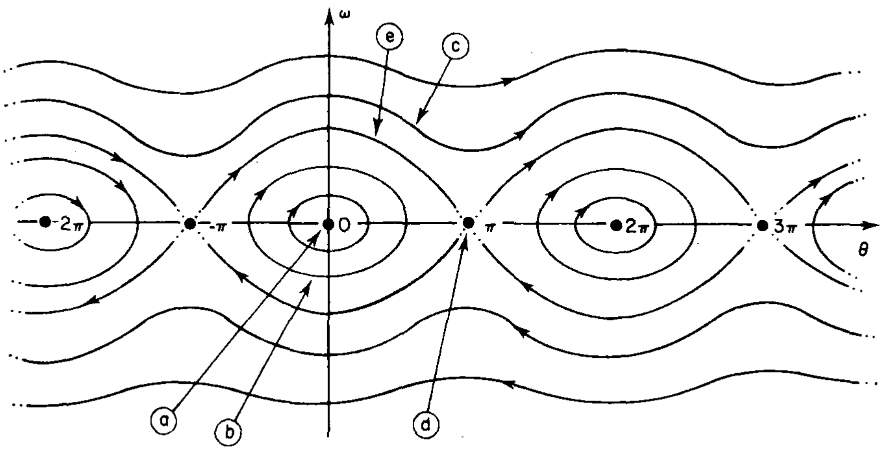

FIGURE 0.4

types of integral curves by their dissimilar appearances. They can be interpreted as follows:

(a) the pendulum hangs vertically downwards and is permanently at rest,
(b) the pendulum swings between two positions of instantaneous rest equally inclined to the vertical,
(c) the pendulum continually rotates in the same direction and is never at rest,
(d) the pendulum stands vertically upwards and is permanently at rest,
(e) the limiting case between (b) and (c), when the pendulum takes an infinitely long time to swing from one upright position to another.

The phase portrait in Figure 0.4 has certain unsatisfactory features. Firstly, the pendulum has only two equilibrium positions, one *stable* (downwards) and one *unstable* (upwards). However, to each of these there correspond infinitely many point curves in the phase portrait. Secondly, solutions of type (c) are periodic motions of the pendulum but appear as nonperiodic curves in the phase portrait. The fact of the matter is that unless we have some very compelling reason to do otherwise we ought to regard $\theta = \theta_0$ and $\theta = \theta_0 + 2\pi$ as giving the same position of the pendulum, since there is no way of instantaneously distinguishing between them. That is to say, the *configuration space*, which is the differentiable manifold representing the spatial positions of the elements of the mechanical system, is really a circle rather than a straight line. To obtain a state space that faithfully describes the system, we replace the first factor $\mathbf{R}$ of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ by the circle S^1, which is the real numbers reduced modulo 2π. Keeping θ and ω as our parameters, we obtain the phase portrait on the cylinder $S^1 \times \mathbf{R}$ shown in Figure 0.5.

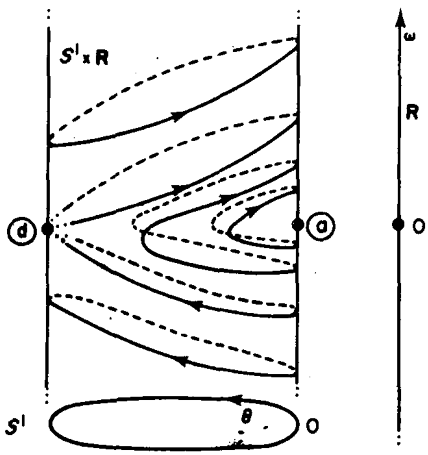

FIGURE 0.5

Consider now the *kinetic energy* T and the *potential energy* V of the pendulum, given by $T(\theta, \omega) = \frac{1}{2}\omega^2$ and $V(\theta, \omega) = g(1 - \cos\theta)$. Writing $E = T + V$ for the *total energy* of the pendulum, we find that equations (0.3) imply that $E' = 0$. That is to say E is constant on any integral curve. In view of this fact, the mechanical system is said to be *conservative* or *Hamiltonian*. In fact, in this example, the phase portrait is most easily constructed by determining the level curves (contours) of E. A pleasant way of picturing the role of E (due to E. C. Zeeman) is to represent the state space cylinder $S^1 \times \mathbf{R}$ as a bent tube in Euclidean 3-space and to interpret E as height. This is illustrated in Figure 0.6. The two arms of the tube contain solutions

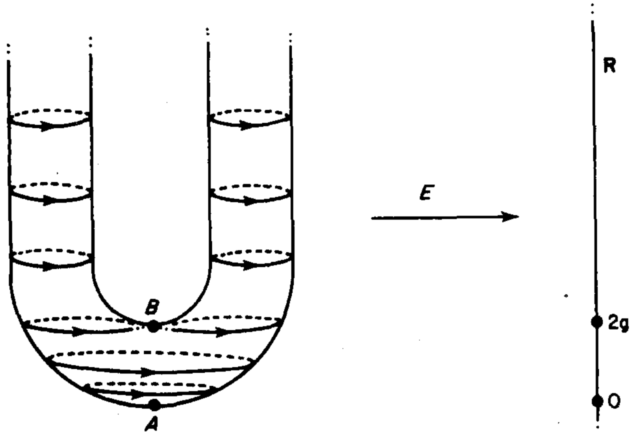

FIGURE 0.6

corresponding to rotations of the pendulum in opposite directions with the same energy E, with $E > 2g$, the potential energy of the unstable equilibrium.

The stability properties of individual solutions are apparent from the above picture. In particular, any integral curve through a point that is close to the stable equilibrium position A remains close to A at all times. On the other hand, there are points arbitrarily close to the unstable equilibrium position B such that integral curves through them depart from a given small neighbourhood of B. Note that the energy function E attains its absolute minimum at A and is stationary at B. In fact it has a saddle point at B.

II. A DISSIPATIVE SYSTEM

The conservation of the energy E in the above example was due to the absence of air resistance and of friction at the pivot Q. We now take these forces into account, assuming for simplicity that they are directly proportional to the angular velocity. Thus we replace equation (0.2) by

(0.7)
$$\theta'' = -g \sin \theta - a\theta'$$

for some positive constant a, and (0.3) becomes

(0.8)
$$\theta' = \omega,$$
$$\omega' = -g \sin \theta - a\omega.$$

We now find that $E' = -a\omega^2$ is negative whenever $\omega \neq 0$. Thus the energy is dissipated along any integral curve, and the system is therefore said to be *dissipative*. If, as before, we represent E as a height function, the inequality $E' < 0$ implies that the integral curves cross the (horizontal) contours of E downwards, as shown in Figure 0.9.

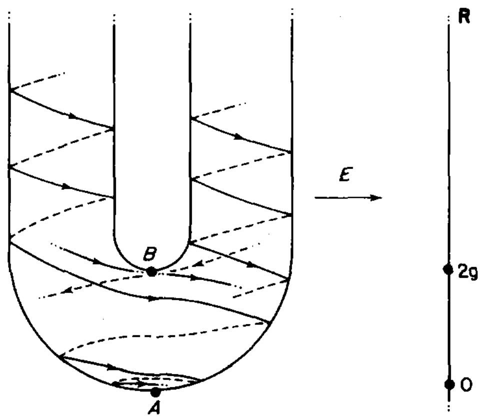

FIGURE 0.9

The reader may care to sketch dissipative versions of Figures 0.4 and 0.5. Notice that the stable equilibrium is now *asymptotically stable*, in that nearby solutions tend towards A as time goes by. We still have the unstable equilibrium B and four strange solutions that either tend towards or away from B. In practice we would not expect to be able to realize any of these solutions, since we could not hope to satisfy the precise initial conditions needed, rather than some nearby ones which do not have the required effect. (One can, in fact, sometimes stand a pendulum on its end, but our model is a poor one in this respect, since it does not take "limiting friction" into account.)

A comparison of the systems of equations (0.3) and (0.8) gives some hint of what is involved in the important notion of *structural stability*. Roughly speaking, a system is structurally stable if the phase portrait remains qualitatively the same when the system is modified by any sufficiently small perturbation of the right-hand sides. By *qualitatively* (or *topologically*) *the same*, we mean that some homeomorphism of the state space maps integral curves of the one onto integral curves of the other. The existence of systems (0.8) shows that the system (0.3) is not structural stable, since the constant a can be as small as we like. To distinguish between the systems (0.3) and (0.8), we observe that most solutions of the former are periodic, whereas the only periodic solutions of the latter are the equilibria. (Obviously this last properly holds in general for any dissipative system, since E decreases along integral curves.) The systems (0.8) are themselves structurally stable, but we do not attempt to prove this fact.

III. THE SPHERICAL PENDULUM

In the case of the simple pendulum, it is desirable, but not essential, to use a state space other than Euclidean space. With more complicated mechanical systems, the need for non-Euclidean state spaces is more urgent; it is often impossible to study them globally using only Euclidean state spaces. We need other spaces on which systems of differential equations can be globally defined, and this is one reason for studying the theory of differentiable manifolds.

Consider, for example, the *spherical pendulum*, which we get from the simple pendulum by removing the restriction that PQ moves in a given plane through Q. Thus P is constrained to lie on a sphere of radius 1 which we may as well take to be the unit sphere $S^2=\{(x, y, z): x^2+y^2+z^2=1\}$ in Euclidean 3-space. We use Euler angles θ and ϕ to parametrize S^2, as in Figure 0.10.

The motion of P is then governed by the second order equations

(0.11)
$$\theta''=\sin\theta\cos\theta(\phi')^2+g\sin\theta,$$
$$\phi''=-2(\cot\theta)\theta'\phi',$$

which we replace by the equivalent system of four first order equations

(0.12)
$$\theta'=\lambda,$$
$$\phi'=\mu,$$
$$\lambda'=\mu^2\sin\theta\cos\theta+g\sin\theta,$$
$$\mu'=-2\lambda\mu\cot\theta.$$

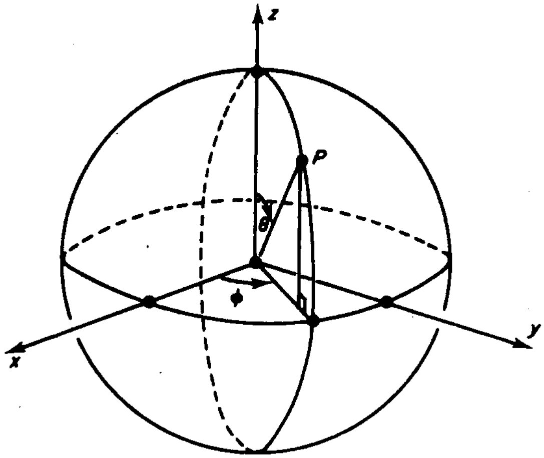

FIGURE 0.10

However, since the parametrization of S^2 by θ and ϕ is not even locally one-to-one at the two poles $(0, 0, \pm 1)$, it is wrong to expect that the four numbers $(\theta, \phi, \lambda, \mu)$ can be used without restriction to parametrize the state space of the system as $\mathbf{R}^4$. They *can* be employed with restrictions (for example $0<\theta<\pi$, $0<\phi<2\pi$) but they do not then give the whole state space. In fact, the state of the system is determined by the position of P on the sphere, together with its velocity, which is specified by a point in the 2-dimensional plane tangent to S^2 at P. The state space is not homeomorphic to $\mathbf{R}^4$, nor even to the product $S^2 \times \mathbf{R}^2$ of the sphere with a plane, but is the tangent bundle TS^2 of S^2. This is the set of all planes tangent to S^2 and it is an example of a non-trivial vector bundle. *Locally*, TS^2 is topologically indistinguishable from $\mathbf{R}^4$, and we can use the four variables θ, ϕ, λ and μ as local coordinates in TS^2, provided that (θ, ϕ) does not represent the north or south pole of S^2.

The system is conservative, so again $E' = 0$ along integral curves, where the energy E is now a real function on TS^2 which, in terms of the above local coordinates, has the form

$$E(\theta, \phi, \lambda, \mu) = \tfrac{1}{2}(\lambda^2 + \mu^2 \sin^2 \theta) + g(1 + \cos \theta).$$

Thus every solution is contained in a contour of E. The contour $E = 0$ is again a single point at which E attains its absolute minimum, corresponding to the pendulum hanging vertically downwards in a position of stable equilibrium. The contour $E = 2g$ again contains the other equilibrium point, where the pendulum stands vertically upright in unstable equilibrium. At this point E is stationary but not minimal. The reader who is acquainted with

Morse theory (see Hirsch [1] and Milnor [3]) will know that for $0<c<2g$ the contour $E^{-1}(c)$ is homeomorphic to S^3, the unit sphere in $\mathbf{R}^4$. In any case, it is not hard to see this by visualizing how the contour is situated in TS^2. For $c>2g$, $E^{-1}(c)$ intersects each tangent plane to S^2 in a circle, and thus can be deformed to the unit circle bundle in TS^2. This can be identified with the topological group $SO(3)$ of orthogonal 3×3 matrices, for (the position vector of) a point of S^2 and a unit tangent vector at this point determine a right-handed orthonormal basis of $\mathbf{R}^3$. Moreover, rather less obviously (see, for example, Proposition 7.12.7 of Husemoller [1]), $SO(3)$ is homeomorphic to real projective space $\mathbf{RP}^3$.

The spherical pendulum is, as a mechanical system, symmetrical about the vertical axis l through the point of suspension O. By this we mean that any possible motion of the pendulum gives another possible motion if we rotate the whole motion about l through some angle k, and that, similarly, we get another possible motion if we reflect it in any plane containing l. This symmetry shows itself in the equations (0.12), for they are unaltered if we replace ϕ by $\phi+k$ or if we replace ϕ and μ by $-\phi$ and $-\mu$. We say that the orthogonal group $O(2)$ acts on the system as a group of symmetries about the axis l. Symmetry of this sort is quite common in mechanical systems, and it can reveal important features of the phase portrait. In this case, for any c with $0<c<2g$, the 3-sphere $E^{-1}(c)$ is partitioned into a family of tori, together with two exceptional circles. The picture that we have in mind is Figure 0.13 rotated about the vertical straight line m. This decomposes $\mathbf{R}^3$ into a family of tori, together with a circle (through p and q) and the line m. Compactifying with a "point at ∞" (see the appendix to Chapter 2) turns $\mathbf{R}^3$ into a topological 3-sphere and the line m into another (topological) circle.

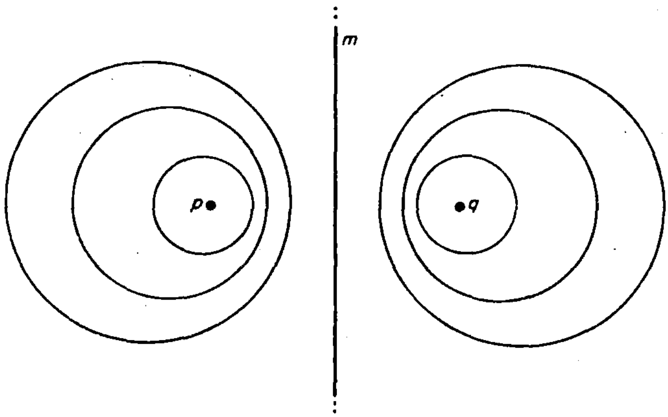

FIGURE 0.13