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PRESENTAZIONE

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Ricordiamo: dapprima, cioè dal I Corso (tenuto nel 1953) fino al IX Corso incluso (tenuto nel 1958), essi furono pubblicati in fascicoli separati del Supplemento *al* Nuovo Cimento *ed editi dalla Casa Nicola Zanichelli di Bologna; indi i Corsi dal X al XIII (tenuti nel 1959) furono pubblicati sotto il titolo di* Rendiconti della Scuola Internazionale di Fisica «Enrico Fermi», *in fascicoli (brochés) indipendenti dal* Supplemento, *editi ugualmente dalla Casa Zanichelli; ora infine dal XIV Corso (tenuto nel 1960) in poi i* Rendiconti *vengono editi in volumi (rilegati) dalla Academic Press.*

Varianti esteriori e varianti editoriali: come si vede. Ma internamente la pubblicazione mantiene, non solo la medesima veste tipografica, ma — ciò che più importa — le medesime caratteristiche avute finora, che sono poi quelle stesse della Scuola: buona scienza, attualità, importanza, scioltezza, critica.

Nove anni fa, terminando il mio discorso inaugurale del I Corso, osavo assegnare alla Scuola l'impresa che fu già del grande Farnese. Oggi, lasciando la Presidenza della Società Italiana di Fisica e quindi consegnando in altre mani anche questa attività che è la Scuola Internazionale di Fisica, mi è gradito ripetere, come auspicio e con immutata fiducia, le parole di quella impresa:

votis subscribent fata secundis.

Milano, 31 Dicembre 1961

GIOVANNI POLVANI

INDICE

Questions of Irreversibility and Ergodicity.

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1. – Introduction.

The aim of the following considerations is to point to some problems, partly of epistemological, partly of more technical character, raised by the concept of irreversibility and its position in the framework of statistical thermodynamics. Opinions concerning such questions are divided; but the origin of the confusion still prevailing in this domain must be sought more in the insufficient care taken by writers on the subject to analyse the epistemological basis of their point of view than in some inherent obscurity of the situation. I shall therefore try in the following to develop one consistent line of argument, which has the advantage of making use of well-established epistemological conceptions, namely those which have proved decisive in the elucidation of the foundations of quantum theory. I hope in this way to provide at least a convenient starting point for further discussion.

I shall first point out that irreversibility is introduced in the description of physical phenomena at a very early stage, and is not directly connected with the degree of complexity of the physical systems under consideration. Then, the well-known dilemma of statistical thermodynamics with respect to irreversibility on the macroscopic scale will be discussed from the point of view of complementarity. Finally, the connexion between macroscopic irreversibility and ergodic properties of mechanical systems will be briefly reviewed.

2. – The « arrow of time ».

It is clear that since the structure of the « elementary » laws of physics (and this includes the laws of quantum theory as well as those of classical mechanics, electromagnetism and gravitation) is essentially reversible, *i.e.* invariant for a reversal of the sense of flow of time, any irreversible behaviour of a physical system must have its origin in some circumstance preventing a complete analysis of this behaviour in terms of such laws.

Within the framework of classical physics it was natural to seek the source of irreversibility in the statistical element introduced in the description of macroscopic systems by the complexity of their atomistic structure. The prevailing view was that there is in principle no limit to a strictly deterministic account of the phenomena, whose evolution in time is necessarily reversible; any observed irreversibility can therefore only be « apparent », in the sense that it refers only to the most probable variation of the average distribution density of the atomic constituents of the system under consideration; the underlying reversibility on the atomic scale is then revealed by the character of the statistical fluctuations around the average variation. Characteristic of this conception is the endeavour to secure objectivity by eliminating all reference to the processes by which we may gain access to the states of the system entering the description of its observed behaviour.

In accordance with this « mechanistic » philosophy, BOLTZMANN [1] suggested that the sense of the « arrow of time » is imposed upon us by the direction of increasing entropy on a cosmical scale in the part of the universe which is accessible to our observation. He imagined that large parts of the universe may have been brought by statistical fluctuation into states far removed from thermal equilibrium: to an observer living in any one of these regions there will be a trend towards equilibrium which will define the sense of time. The lower limit for the time-scale of such an evolution and the size of the region taking part in it give an idea of the magnitude of the kind of fluctuation one has to assume. Moreover, one must realize that if at any time a large fluctuation is present, it is overwhelmingly probable that it has arisen from a still larger one. Even without leaving the ground of XIX-th century mechanicism, therefore, the conjecture of a thermodynamical origin of irreversibility in nature leads to cosmological hypotheses which, to say the least, are not very plausible.

The development of quantum theory, however, has taught us that the whole epistemology of mechanicism is untenable and has to be replaced by another conception, embodying a deeper-going analysis of the relation between the description of the phenomena and their observation. Although the role of the process of observation for the very definition of the concepts in terms of which the physical laws are formulated has been emphasized in the discussion of typical quantum effects, it is clearly of universal significance and just as imperative, from the epistemological point of view, in classical physics, even though in this domain its quantitative implications may be negligible. Now, it turns out that this analysis of the observation process also throws light on the question of « time's arrow ».

Every observation of a physical system involves an interaction between this system and the apparatus which fixes the experimental conditions and thereby defines the phenomenon which is observed, or the physical quantity which is measured in the process. The observation or measuring process is closed by the registration of some permanent mark left by the system on the appropriate part of the apparatus; this may be a spot on a photographic plate, a stylus tracing or some similar record. Owing to the finite velocity of propagation of all interaction, it is always possible, in principle, to establish a definite order in the time-sequence of the physical process under investigation and its recording by the apparatus; and this offers a natural basis for the definition of the « earlier-later » relationship. In plain words, if we look at an object, we say—by definition, or convention—that a photon which strikes our retina has been emitted *earlier* by the object. Thus, the sense of time is referred to the propagation of « signals » from the object of observation to the detectors by which we observe it (*).

The preceding analysis stresses the role played by a property of the registering device which may be called « memory »: the production of a permanent mark. (*i.e.* a mark whose duration is at least of the order of physiological relaxation times). This memory has no immediate relation (except for the lower time limit just indicated) with that of a human observer, although the ultimate prototype of the observation process is of course the direct sense perception and its recording in the brain. This physiological aspect of the matter, however, is by no means involved in our argument, which stops at the purely automatic physical recording of the phenomenon and thereby makes it clear that our fixation of the direction of time fulfils every reasonable requirement of objectivity. One must take account, in this connexion, of the alteration of our conception of objectivity which results from the recognition of the necessity of including a complete specification of the conditions of observation in the definition of the phenomena.

We further realize that the existence of a definite sense of time, or time-directedness, is a universal character of all physical processes, and in particular independent of the nature of the system considered, which may just as well be a single atomic object as a macroscopic body. There is always, however, an element of macroscopic complexity in this time-directedness, inasmuch as the registering device, in order to provide unique determinations of the quantities under consideration, must be describable in classical terms.

(*) This argument has been developed by NIELS BOHR many years ago. I had recently occasion to present it in a course of lectures [2] and in a discussion [3]. See also a previous article by v. WEIZSÄCKER [4], whose philosophical comments, however, especially regarding the alleged subjectivity of quantum theory, are quite at variance with Bohr's own views.

To that extent, the irreversibility implied in time-directedness is again of statistical character; it shares this character with all concepts directly referring to our immediate experience and used without further analysis in the description of the phenomena; but this does not mean that one could contemplate any return to the thermodynamical arguments outlined above. Indeed, if one would try to analyse the thermodynamical behaviour of the registering device in its interaction with the observed system, in the hope of thereby obtaining further insight into the irreversibility of the recording process, this would amount to treating the registering device as a physical system under observation, whereby it would no longer fulfil its intended function of providing a *unique* specification of the phenomenon.

In fact, no amount of shifting the distinction between system observed and measuring apparatus can ever elude the necessity of maintaining this fundamental distinction, by which alone a connexion is established between the system and the concepts by which it is described. The concept of time-directedness appears therefore as a primary one, in the sense that it is immediately related to the process of observation, at a deeper epistemological level than the type of statistical irreversibility occurring in thermodynamics. The latter can be analysed in its own right (with the help of the H-theorem and the consideration of the fluctuations, as outlined above), but it must be realized that in this analysis the sense of time is already determined independently of any statistical consideration. The statement that the entropy of an isolated system in the final state of any process is larger than in the initial state has accordingly in *every* case a well-defined physical content, and can never be regarded as an indirect definition of the sequence « initial-final ». Such a conclusion is in harmony with the fact that any irreversible change in the state of an isolated system must be initiated by an external intervention: the removal (without expenditure of work) of some constraint, which can only lead to a state which is less determined and therefore has a larger entropy. GIBBS had no doubt this situation in mind when he wrote [5]: « ... while the distinction of prior and subsequent events may be immaterial with respect to mathematical fictions, it is quite otherwise with respect to the events of the real world. » In fact, he refers the time-directedness in the « real world » to an essential difference in our estimation of probabilities of subsequent and anterior events, which he describes in the at first sight sybilline sentence: « while the probabilities of subsequent events may often be determined from the probabilities of prior events, it is rarely the case that probabilities of prior events can be determined from those of subsequent events, for we are rarely justified in excluding the consideration of the antecedent probability of the prior events. »

The fundamental significance of time-directedness, in the sense defined above, is illustrated by the fact, pointed out by KRONIG [6] many years ago, that it is equivalent to the existence of dispersion relations for the Fourier amplitudes of the field whose propagation is considered: a function satisfying dispersion relations

$$f(\omega) = \frac{1}{\pi i} \mathrm{P}\int_{-\infty}^{\infty} \frac{f(\omega')\, d\omega'}{\omega' - \omega}$$

is in fact the Fourier transform of a function vanishing for negative values of its argument, and which therefore (if its argument is the time variable) may represent the propagation of a signal. In the recent development of the theory of dispersion relations, such functions are usually described as obeying a « causality condition »; this terminology is unfortunate since causal relations are time-symmetrical; what is actually meant is that the functions in question express the time-directedness of the field propagation (*).

3. – Complementarity of dynamical and statistical descriptions.

The analysis of macroscopic irreversibility on the basis of the H-theorem has resolved the apparent contradiction between this type of behaviour and the reversibility of the underlying dynamical laws of time evolution by introducing a statistical element into the passage from the atomistic to the macroscopic picture. In this way, the two modes of description—dynamical and statistical—are incorporated into a synthetic account as two complementary aspects of the behaviour of large assemblies of atoms (**). This situation is even an especially simple and instructive example of the peculiar kind of logical relationship to which the name of complementarity may be applied.

Above all, it is characterized by the mutual exclusiveness of the two descriptions: conditions allowing of a complete mechanical (and electrodynamical) description of a system exclude the possibility of applying to the system any of the typical thermodynamical concepts; and conversely, the definition of the latter requires conditions of observation under which the mechanical parameters essentially escape our control. Thus, in order to assign to the system a definite temperature, it is necessary to allow it to exchange energy with a thermostat, and it is thereby impossible to assign to its energy any definite value; conversely, in order to keep its energy at a constant value, one must isolate the system, and the concept of temperature cannot be applied to it.

(*) Cfr. on this point ref. [2].

(**) This was first pointed out by NIELS BOHR in his Faraday lecture [7] in 1932. See also refs. [8] and [9].

This complementarity is often overlooked owing to the fact that for very large systems it is permissible to disregard the fluctuations of the macroscopic quantities around their averages, and, for practical purposes, to assimilate a canonical ensemble of given modulus to the microcanonical ensemble corresponding to the average energy. Such a procedure may be compared to the disregard of the quantal limitations imposed by the uncertainty relations when one is dealing with phenomena involving many quanta of action. A closer examination of the statistical fluctuations, however, reveals reciprocal relationships between them, closely analogous to the uncertainty relations of quantum theory.

Let us consider a small part (but still of macroscopic dimensions) of a very large system, and study the deviation of this part-system from equilibrium, characterized by changes of the thermal variables, entropy S and temperature T, and of a typical pair of other macroscopic quantities, consisting of an extensive variable a and an intensive variable A, specified by their contribution $-A\,da$ to the change of the total energy. The (root mean square) fluctuation of all thermodynamical quantities pertaining to the part-system, in such random deviations from equilibrium, may be calculated by a method due to LANDAU and LIFSCHITZ [10]. These authors show that the fluctuations of the pairs of variables (T, a) and (S, A) are statistically independent, and that (k denoting Boltzmann's constant)

$$(1)\qquad \begin{cases} \langle \Delta T^2 \rangle = kT^2 C_a^{-1}, & \langle \Delta a^2 \rangle = kT D_T ; \\ \langle \Delta S^2 \rangle = kC_A , & \langle \Delta A^2 \rangle = kT D_S^{-1} . \end{cases}$$

In these formulae, C_a, C_A represent the heat capacities at constant a or A, and D_T, D_S the absolute values of the isothermal and adiabatic rates of change of a on variation of A:

$$(2)\qquad D_T = -\left(\frac{\partial a}{\partial A}\right)_T, \qquad D_S = -\left(\frac{\partial a}{\partial A}\right)_S ;$$

all these factors C and D are extensive, *i.e.* (for a homogeneous body) proportional to the average number N of atoms (or other elementary constituents) of the part-system.

From eqs. (1) it is therefore apparent that the fluctuations of extensive variables are proportional to $N^{\frac{1}{2}}$, whereas those of intensive variables are proportional to the inverse of this quantity. In particular, if we assume for simplicity that the volume of the part-system is invariable, the square of the energy fluctuation will be

$$(3)\qquad \langle \Delta E^2 \rangle = kT^2 C_a ,$$

and it is clear that we cannot simultaneously reduce the fluctuations of both temperature and energy of the part-system by increasing its size. In fact the product of the two fluctuations is independent of this size:

$$\langle \Delta E^2 \rangle^{\frac{1}{2}} \langle \Delta T^2 \rangle^{\frac{1}{2}} = kT^2 \, . \tag{4}$$

For a given temperature (defined by the very large total system), the order of magnitude of this product is governed by the universal constant k, or the inverse of Avogadro's number; this is the atomistic parameter which plays a part similar to the quantum of action in fixing the quantitative importance of reciprocal limitations in the use of different modes of description.

In the present case it is of course the size, more precisely the number of degrees of freedom, of the system which is decisive for the applicability of either dynamical or thermodynamical concepts to the analysis of its behaviour; in particular, the Landau-Lifschitz formulae (1) clearly show that no definite temperature or pressure can be assigned to a system of only a few degrees of freedom. There is no point in saying that the statistical definition of these concepts is still formally valid for such systems: for the smallness of the fluctuations is an essential condition for the practical usefulness of statistically defined quantities (*). Here, as always, we find that the character of the physical description is directly imposed by the conditions of observation: the statistical character of the thermodynamical description arises from the large number of degrees of freedom of macroscopic systems, and its meaningfulness is accordingly limited to such systems; the preceding discussion of fluctuations has no other aim than to illustrate this situation by more quantitative considerations.

It may be instructive to pursue the matter somewhat further, and consider one of those intermediate cases in which one is dealing with a system, like an emulsion droplet, large enough to be accessible to direct individual observation, and yet small enough to exhibit large fluctuations. Such an object may then be considered from either of the complementary points of view, and the two corresponding modes of description may be applied to it with reciprocal limitations: we may on the one hand ascribe to this body an individual motion, and on the other hand treat it as a member of an assembly characterized by a statistical distribution and the resulting thermodynamical quantities. However, the motion will not have a purely dynamical aspect: it will be a « Brownian » motion; neither will the statistical distribution be so well defined as the ideal ones which correspond to systems with very large numbers of constituents. To the translational motion of the barycentre of the droplet

(*) In a recent paper by J. Blatt [11], this point is not sufficiently taken into account.

there correspond an intensive variable, the velocity v, and an extensive one, the momentum p; they are of course connected by the relation $p = Mv$ if M denotes the mass of the droplet, and their contribution to the energy differential is $v\,dp$. Application of eqs. (1) to this case gives, in particular,

$$\langle \Delta S^2 \rangle^{\frac{1}{2}} \langle \Delta v^2 \rangle^{\frac{1}{2}} = k v_0 \,, \tag{5}$$

where

$$v_0 = \sqrt{\frac{CT}{M}} \tag{6}$$

is of the order of the mean atomic velocities inside the droplet. The formula (5) expresses the reciprocal limitation (independent of the size of the droplet) of the statistical and dynamical descriptions of its behaviour.

In spite of the widegoing parallelism between the complementary situations in statistical thermodynamics and quantum theory, people are often reluctant to admit that the two cases are equally fundamental: while in quantum theory one is clearly confronted with the inescapable necessity of using two complementary sets of concepts in order to achieve an exhaustive account of experience, it would seem that in thermodynamics the dynamical description, applied to the systems of atoms constituting the bodies under investigation, is sufficient in all cases to account for the observed phenomena: the proper thermodynamical quantities would merely appear as convenient auxiliaries, derived by statistical procedures from the dynamical quantities; no doubt, their use might be subject to restrictions, but these would not have the same radical character as the quantal limitations expressed by the uncertainty relations.

The flaw in such an argument is that it does not sufficiently take into consideration the peculiar epistemological problems arising from the fact that all our knowledge about the atomic constitution of matter and the behaviour of atomic systems is derived from macroscopic observations and can only be formulated in terms of concepts referring to such observations. The kind of conceptual difficulty arising from this fact is illustrated by the necessity, when discussing the dynamical description at the atomic level, of appealing to fictitious observers, who are called « demons » precisely because they are required to perform feats incompatible with the limitations of human observation. The concepts of thermodynamics have been constructed, historically, to cope with a definite situation characterized by a certain type of macroscopic observations; as Clausius said, they are « not concerned with what heat can do with the help of demons, but with what it can do by itself ». The dynamical mode of description applies to phenomena defined by different conditions of macroscopic observation which are in a relation of mutual exclusion, or complementarity, to those of thermodynamics. This is a perfectly objective relationship, reflecting different modes of behaviour, under different external conditions, of systems composed of a large number of atomic constituents; there can therefore be no question of eliminating it. The logical role of the atomistic picture underlying statistical thermodynamics is rather, by providing a common foundation for the two complementary aspects, to allow a more precise determination of the conditions under which they appear.

It is instructive to reflect that we are facing an entirely similar epistemological situation in biology [12]: here there is a relationship of complementarity between the mode of description of molecular biology, in which the processes of life are accounted for in terms of ordinary physical and chemical concepts and laws, and that of macroscopic physiology, based on the concepts of biological function and organization. Here again, we find the two complementary aspects distinguished by different types of causality: in molecular biology we have just the statistical causality characteristic of quantum theory, whereas concepts like function and organization imply a consideration of final causes —not, of course, with any suggestion of an underlying finality arising from some principle fundamentally different from those governing inanimate matter, but simply in a formal methodological sense. Just as the atomistic conception is the common basis of the two complementary pictures of thermal phenomena, the structural properties of macromolecules, entirely accounted for by the known laws of quantum chemistry, are a sufficient basis for the understanding of all biological processes (at any rate, there is in our present experience no indication of any possible limitation in this respect). This, however, does not mean that the traditional functional approach to these processes would thereby become superfluous; not primarily because the excessive complexity of most living organisms prevents a detailed analysis of their behaviour at the macromolecular level from being actually performed, but rather on purely logical grounds: it is quite obvious that without the formal concept of some organization adapted to certain functions, or even several such concepts, corresponding to the successive stages of increasing complexity of biological evolution [13], it is impossible to bring order into the wealth of structures and behaviours which is the most striking character of life. By emphasizing the complementarity of the two contrasting points of view one preserves the possibility of doing equal justice, in a rational way, on the one hand to the unity of molecular structure and behaviour underlying all biological processes, and on the other to the qualitative diversity they exhibit at the same time.

We have thus surveyed three great subdivisions of a general account of natural phenomena, corresponding to increasing complexity or specialization of structure: quantum theory, which deals with the simplest atomic systems, statistical thermodynamics, whose main concern is the atomic constitution of matter in bulk, and biology, which describes the properties of peculiar macromolecular structures and the resulting macroscopic aggregations, which present characters of organization of increasing refinement. At each stage, we meet with a typical complementarity between mutually exclusive modes of description, the origin of which is ultimately the same: the necessity of applying essentially macroscopic concepts to the account of experience concerning processes at the atomic level. This necessity is probably very deeply rooted in the evolution of living organisms, for it is difficult to believe that the communication system which we call thinking could develop in an organ of less complexity than the human brain.

4. – Irreversibility and ergodicity.

The physical basis of the atomistic interpretation of macroscopic irreversibility within the framework of classical mechanics was laid in all essentials by Boltzmann's conception of the ergodic character of the trajectories in phase space representing the time evolution of a large mechanical system. This idea was most instructively visualized by Gibbs' famous simile of the mixture of two liquids of different colours, which at the same time—as Ehrenfest especially emphasized—illustrated the fact that all statements of a statistical nature (and thereby exhibiting irreversible features) refer to distributions defined by a *coarse-grained* density in phase space. A rigorous mathematical formulation of these conceptions had to wait for the elaboration of appropriate methods, but was eventually achieved on the basis of the concept of metric transitivity.

The extension of the argument to quantum theory raises a delicate problem: whereas in classical theory the coarse-grainedness of the subdivision of phase space symbolizing the macroscopic specification of the state of the system does not in any way affect the functional representation of the physical quantities, this is no longer so in quantum theory, owing to the uncertainty relations between canonically conjugate mechanical variables. A general method for representing coarse-grained distributions of quantal systems, and for defining the corresponding « coarse » operators, was proposed by J. VON NEUMANN [14]. With its help, he was able to give a quantal treatment of the mixing process presenting a wide-going analogy to the classical discussion (cf. on this point ref. [9]). However, it proved impossible to eliminate completely from the result all reference to the structure of the coarse distribution: the asymptotic statistical distribution could only be established in the sense of probability, and the proof accordingly involved, in its last stage, an averaging over all possible coarse distributions.

A few years ago, FIERZ [15] raised doubts about the adequacy of this averaging process, and his paper stimulated others to a critical re-examination of the problem. It soon became clear that if the averaging was performed from the start (and not as the last stage of the argument, as in von Neumann's proof), every reference to the structure of the system, such as entailed by ergodicity conditions [16], and even to its time evolution [17], was completely swept away from the resulting statistical properties; the latter, which we may call « pseudo-ergodic », were therefore too general to be of any physical significance (*). It must be emphasized, however, that neither Fierz nor the other critics have brought forward any direct objection to von Neumann's original approach, which in my opinion still stands as an acceptable foundation for the statistics of quantal systems.

Nevertheless, the critical investigations just mentioned, in which an Italian group of physicists had taken a prominent part, had the happy consequence of prompting these physicists to explore new possibilities, which led them to interesting results (*). They [20,21] in fact discovered a class of statistical inequalities very similar in form to the pseudo-ergodic ones derived by an all-too-sweeping application of von Neumann's average, but quite different in mathematical character as well as physical meaning. In this approach, no average is taken over the coarse distribution; instead, one considers statistical averages over the initial-state vectors of the system, specified by the introduction of a definite measure in Hilbert space; and the derivation of the inequalities in question does require that the time evolution of the system be governed by a unitary operator. These inequalities reduce asymptotically to the desired statement of the mixing process when the number of degrees of freedom of the system is sufficiently large. Clearly, no ergodicity conditions are implied in such considerations since the possibility of exceptional behaviour is allowed for by the mathematical formulation; nevertheless, it is easily shown that the approach to the canonical distribution for any system coupled to a thermostat is ensured as a consequence of this coupling. It is therefore apparent that we have here an alternative treatment of the fundamental problem of statistical thermodynamics which is equally satisfying from the mathematical and the physical point of view. It is further instructive to note, as has been pointed out by the Italian physicists [22], that this treatment can be closely paralleled in classical theory, on the basis of a remark originally made by KHINCHIN. There are thus at the moment two rival modes of analysis of the statistical foundations of thermodynamics; we shall now take up in turn the main points of the general argument and examine in what light they appear from these two points of view.

(*) This critical phase, as well as the kinetic aspects not touched upon here, are surveyed in ref. [18]. For the subsequent developments and further references to the whole problem, see ref. [19].

4·1. *The general argument.* – It would hardly be necessary to recall the well-known chain of reasoning which leads to the definition of the concepts of temperature and associated canonical distribution (cf. *e.g.* ref. [9]), if it were not for the fact that ever recurring criticisms levelled against it on insufficient grounds invite some brief comment. It will be convenient to distinguish three steps in the argument:

1) Birkhoff's theorem for isolated systems (and its quantal analogue);

2) the introduction of a coarse-grained density distribution and the « mixing process » leading to the asymptotic equilibrium distribution;

3) the derivation of the canonical distribution for a system weakly coupled to a very large « thermostat ».

In the third step, the total system formed by the given system and the thermostat is treated as isolated, and it then appears that the establishment of the canonical distribution for the given system is a direct consequence of the mere existence of a coupling, however weak, between this system and the thermostat; the coarse-grainedness of the (microcanonical) distribution of the total system has no influence on the result since it concerns only the thermostat variables. These simple features of an analysis going back to Boltzmann suffice to show how unfounded are the doubts recently expressed by Blatt [11] about the adequacy of the idealizations of isolated system and coarse-grained distribution.

The notion of isolated system is an abstraction of exactly the same type as any other physical concept, and its use in a given situation is merely a matter of practical convenience. In a mode of exposition adapted as closely as possible to the physical situation, it is the canonical distribution, and not the microcanonical one, which forms the natural basis for the statistical interpretation of thermodynamics. The necessity of a lack of isolation for bringing about thermal equilibrium, on which both Blatt [11] and Fierz [15] insist, is indeed directly illustrated by the role of the coupling to the thermostat in the establishment of the canonical distribution; but it can in no way be regarded as a valid objection to the consideration of isolated systems as an auxiliary step in the mathematical derivation of results with direct physical application.

Likewise, the idea of coarse-grained density is a useful mathematical device to describe in a simple and efficient way how incomplete determination of the variables defining the state of the system affects statements which can be made about this state. Since statements of this type about the « mixing process » which leads to the microcanonical distribution do not require any other presupposition than the law of time evolution of an isolated system, there ought not to be any qualm about their objective character; confusion in this