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Edited by A. Dold and B. Eckmann

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PREFACE

This book presents results arising from several areas of the theory of functions and mathematical physics.

The first of these sources, the theory of generalized analytic (pseudo-analytic) functions of L. Bers [a,b] and I.N. Vekua [a,b] has been constructed within the framework of a general interest in different generalizations of analyticity. It was established that such fundamental properties of analytic functions as the argument principle, the Liouville theorem and so on are inherent in solutions of all linear elliptic systems of first order with two unknown functions on the plane. By quasiconformal mappings these systems can be reduced to the complex Carleman-Bers-Vekua equation

$$\bar{\partial} u + au + b\bar{u} = 0 \tag{1}$$

Later the theory of matrix equations (1) was built (W. Wendland [a]). These equations are extremely important for applications (see §12).

At the same time on Riemann surfaces the Riemann boundary problem

$$F^{+}(p) = G(p)\ F^{-}(p) \tag{2}$$

was studied (A. Grothendieck [a], W. Koppelman [b,c], Yu.L. Rodin [a,c,p], H. Röhrl [a,b] and other authors). Main facts of the algebraic function theory were related with the theory of singular integral operators and the classification problem of vector bundles over Riemann surfaces. Afterwards this theory found fundamental physical applications (the Riemann problem method of V.E. Zakharov - A.B. Shabat) in the inverse scattering problem, the integrable systems theory and the solitons theory. At last, recently generalized analytic functions were used in these areas too (see M.J. Ablowitz, D. Bar Yaacov, A.S. Fokas [a], A.S. Fokas, M.J. Ablowitz [a,b], I.M. Krichever, S.P. Novikov [a], A.V. Mikhailov [a,b], V.E. Zakharov, S.V. Manakov [a], V.E. Zakharov, A.V. Mikhailov [a]).

These circumstance stimulated the study of generalized analytic functions on Riemann surfaces. The work was begun by L. Bers [c] and was continued by W. Koppelman [c] and the author [d-j,l]. In our book this area is presented systematically for the first time.

Chapter 1 is devoted to the Riemann-Roch theorem and, naturally, is enclosed into the general theory of the index of elliptic operators with corresponding simplifications. In Chapter 2 multivalued solutions of equation (1) are studied. It demands to look for some representations of them. In particular, the methods allow to obtain a direct proof of the Riemann-Roch theorem. In Chapter 5 they are used to study singular cases and surfaces of infinite genus. In Chapter 3 we expound the Riemann boundary problem and its connections with the Riemann-Roch and the Abel theorems, the Jacobi inversion problem and the classification problem for bundles. The main and most difficult problem of generalized analytic function theory is solved in Chapter 4. It is known that the Abel problem of the existence of an analytic function with prescribed zeros and poles on a compact Riemann surface cannot be solved by pure algebraic methods and demands to use a transcendental operation - applying the logarithm. In our case it leads to a nonlinear integral equation. This equation has been a success to investigate the problem completely.

At last, in §12 we describe very briefly some approaches to physical applications. This is a subject of the expository paper of the author which was published in the journal "Physica D" recently.

The book is addressed to mathematicians and physicists, specialists in the theory of functions, differential equations and mathematical physics (field theory, solitions theory and so on). A preliminary knowledge of the theory of Riemann surfaces and algebraic topology is not necessary for reading the book.

The author is sincerely thankful to his tutor professor L.I. Volkoviskii. He was the initiator for the study of the Riemann problem and generalized analytic functions on Riemann surfaces in the USSR and directed the author's work during many years. The author is glad to express his gratitude to Prof. V.E. Zakharov and Prof. A.V. Mikhailov for numerous fruitful discussions on physical applications of Riemann surfaces. The author is also grateful to Prof. Dr. W.L. Wendland whose moral support was decisive and to Prof. Dr. H. Begehr who edited the manuscript and inserted a number of improvements.

C O N T E N T S

CHAPTER 1

THE DOLBEAULT AND RIEMANN-ROCH THEOREMS

§ 1. Generalized analytic functions in the disk

A. The operator T

1. Consider the Cauchy-Riemann equations

$$\begin{pmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 . \tag{1.1}$$

Letting $u = \varphi + i\psi$ and introducing the operators of complex differentiation

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{1.2}$$

$$z = x + iy$$

we rewrite equation (1.1) in the form

$$\bar{\partial} u = 0 . \tag{1.3}$$

The corresponding inhomogeneous equation has the form

$$\bar{\partial} u = f . \tag{1.4}$$

Let G be a bounded domain of the complex z-plane with a sufficiently smooth boundary ∂G, $\bar{G}$ be its closure and f be a function continuous in G. Then the general solution of (1.4) has the form

$$u(z) = F(z) - \frac{1}{\pi}\iint_G \frac{f(t)\,d\sigma_t}{t-z}, \tag{1.5}$$

$$d\sigma_t = d\xi d\eta , \quad t = \xi + i\eta .$$

Here $F(z)$ is an arbitrary analytic function in G.

We use the Green formulae in the form

$$\iint_G \frac{\partial g}{\partial \bar{z}}\, d\sigma_z = \frac{1}{2i} \int_{\partial G} g\,dz ,$$

$$\iint_G \frac{\partial g}{\partial z}\, d\sigma_z = -\frac{1}{2i} \int_{\partial G} g\,d\bar{z} . \tag{1.6}$$

Then, for any function of the class C^1 in the closed domain $\bar{G}$ the well-known formula

$$u(z) = -\frac{1}{\pi} \iint_G \frac{\partial u}{\partial \bar{t}} \frac{d\sigma_t}{t-z} + \frac{1}{2\pi i} \int_{\partial G} \frac{u(\tau)\,d\tau}{\tau - z} \tag{1.7}$$

is valid. Equation (1.7) involves (1.5).

2. Below, (1.4) will be considered for more weak assumptions. In order to make sure all these formulae are valid for wider function classes, we describe properties of the operator

$$Tf(z) = -\frac{1}{\pi} \iint_G \frac{f(t)\,d\sigma_t}{t-z} . \tag{1.8}$$

This operator belongs to the class of operators of the potential type (A. Calderon, A. Zygmund [a]). We list the properties of the operator T following I.N. Vekua [a].

First some Banach spaces are introduced which will be used below. Let a function $f(z)$ satisfy the Hölder condition

$$|f(z_1) - f(z_2)| \le H\,|z_1 - z_2|^{\alpha} , \qquad 0 < \alpha \le 1 , \tag{1.9}$$

in the closed domain $\bar{G}$. Denote

$$H(f) = \inf H = \sup_{z_1, z_2 \in \bar{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{\alpha}} .$$

Introduce the Banach space $C_\alpha(\bar{G})$ of functions satisfying the Hölder condition with exponent α in $\bar{G}$ with the norm

$$\| f \|_{C_\alpha(\bar{G})} = \max_{z \in \bar{G}} |f(z)| + H(f) = \| f \|_{C(\bar{G})} + H(f) . \tag{1.10}$$

Let $f \in L_p(\bar{G})$, $0 < \alpha \le 1$, be some constant and

$$B(f) = \sup \frac{1}{|\Delta z|^{\alpha}} \| f(z+\Delta z) - f(z) \|_{L_p(\bar{G})} .$$

Introduce the Banach space $L_{p,\alpha}(\bar{G})$ of functions satisfying the inequality

$$\| f(z+\Delta z) - f(z) \|_{L_p(\bar{G})} \le B(f) \; |\Delta z|^{\alpha} \tag{1.11}$$

with the norm

$$\| f \|_{L_p^{\alpha}(\bar{G})} = \| f \|_{L_p(\bar{G})} + B(f) . \tag{1.12}$$

The set of functions continuous in $\bar{G}$ together with their partial derivatives up to the order m inclusive forms the Banach space $C_m(\bar{G})$ with the norm

$$\| f \|_{C_m(\bar{G})} = \sum_{k=0}^{m} \sum_{\ell=0}^{k} \max_{z\in\bar{G}} \left| \frac{\partial^k f}{\partial z^{k-\ell} \, \partial \bar{z}^{\ell}} \right| . \tag{1.13}$$

If all partial derivatives satisfy the Hölder condition, we obtain the space $C_{m,\alpha}(\bar{G})$ with the norm

$$\| f \|_{C_{m,\alpha}(\bar{G})} = \sum_{k=0}^{m} \sum_{\ell=0}^{k} \left\{ \left\| \frac{\partial^k f}{\partial z^{k-\ell} \, \partial \bar{z}^{\ell}} \right\|_{C(\bar{G})} + H\left(\frac{\partial^k f}{\partial z^{k-\ell} \, \partial \bar{z}^{\ell}} \right) \right\} . \tag{1.14}$$

<u>Theorem 1.1</u>. Let $f \in L_p(\bar{G})$, $p > 2$, and

$$g(z) = Tf(z) = -\frac{1}{\pi} \iint_G \frac{f(t)\, d\sigma_t}{t-z} .$$

Then the following estimations are valid:

$$|g(z)| \le M_1 \| f \|_{L_p(\bar{G})} ,$$

$$|g(z_1) - g(z_2)| \le M_2 \| f \|_{L_p(\bar{G})} |z_1 - z_2|^{\alpha} , \quad \alpha = \frac{p-2}{p} . \tag{1.15}$$

Hence the linear operator

$$T: L_p(\bar{G}) \to C_{\alpha}(\bar{G}) , \; \alpha = \frac{p-2}{p} , \; p > 2 ,$$

is compact and

$$\| Tf \|_{C_\alpha(\bar{G})} \le M \| f \|_{L_p(\bar{G})} . \qquad (1.16)$$

Theorem 1.2. If $f \in L_p(\bar{G})$, $1 \le p \le 2$, then the function $g(z) = T f(z)$ belongs to the space $L_{\gamma,\alpha}(\bar{G})$, where γ is an arbitrary number satisfying the inequality

$$1 < \gamma < \frac{2p}{2-p} . \qquad (1.17)$$

Moreover, the following estimations are valid,

$$\| Tf \|_{L_\gamma(\bar{G})} \le M_{p,\gamma} \| f \|_{L_p(\bar{G})} ,$$

$$\Big(\iint_G |g(z+\Delta z) - g(z)|^\gamma \, d\sigma_z\Big)^{1/\gamma} \le M'_{p,\gamma} \| f \|_{L_p(\bar{G})} |\Delta z|^\alpha , \qquad (1.18)$$

$$\alpha = \frac{1}{\gamma} - \frac{2-p}{2p} > 0 .$$

This result entails the complete continuity of the operator T mapping

$$T: L_p(\bar{G}) \to L_{\gamma,\alpha}(\bar{G}) , \quad 1 \le p \le 2 , \quad \alpha = \frac{1}{\gamma} - \frac{1}{p} + \frac{1}{2} , \quad p \le \gamma < \frac{2p}{2-p} .$$

In the following we understand derivatives in the generalized sense. The linear set of functions belonging to $C_m(\bar{G})$ and having compact support in G is denoted by $C_m^0(G)$.

Definition. Let $f,g \in L_1(G)$ and satisfy the relation

$$\iint_G g \frac{\partial \varphi}{\partial \bar{z}} \, d\sigma_z + \iint_G f \varphi \, d\sigma_z = 0 \qquad (1.19)$$

$$\Big(\iint_G g \frac{\partial \varphi}{\partial z} \, d\sigma_z + \iint_G f \varphi \, d\sigma_z = 0 , \qquad (1.19')$$

respectively$\Big)$ for an arbitrary function $\varphi \in C_1^0(G)$. Then the function f is said to be the generalized derivative of g with respect to $\bar{z}$ (with respect to z , respectively)

$$f = \frac{\partial g}{\partial \bar{z}} \qquad \Big(f = \frac{\partial g}{\partial z}\Big) .$$

The class of functions possessing generalized derivatives with respect to $\bar{z}$ is denoted by $D_{\bar{z}}(G)$ ($D_z(G)$, respectively).

The class of functions possessing generalized derivatives belonging to L_p is denoted by $D_{1,p}(G)$. The Banach space of functions possessing generalized derivatives of order $\le m$ with the norm

$$\| f \|_{D_{m,p}(G)} = \sum_{\ell,k=0}^{\ell+k\le m} \left\| \frac{\partial^{\ell+k} f}{\partial z^{\ell}\, \partial \bar{z}^{k}} \right\|_{L_p(\bar{G})} \tag{1.20}$$

is denoted by $D_{m,p}(G)$.

In the case the derivatives are integrable in the closed domain $\bar{G}$ we write $D_{m,p}(\bar{G})$.

Theorem 1.3. If $f = \partial_{\bar{z}} g \in L_1(\bar{G})$, then

$$g(z) = \phi(z) - \frac{1}{\pi} \iint_G \frac{f(t)\,d\sigma_t}{t-z} \tag{1.21}$$

where ϕ is a holomorphic function in G . Conversely, if $\phi(z)$ is a holomorphic function in G and $f \in L_1(\bar{G})$, then the function $g(z) = \phi(z) + T f(z) \in D_{\bar{z}}(G)$ and $\bar{\partial} g = f$. If $u(z) \in C(\bar{G})$ and $\partial_{\bar{z}} u \in L_p(\bar{G})$, $p > 2$, then equation (1.7) is valid.

Theorem 1.4. Let $f(z) \in C_{m,\alpha}(\bar{G})$, $0 < \alpha < 1$, $m \ge 0$. Then the function $g(z) = T f(z)$ belongs to the class $C_{m+1,\alpha}(\bar{G})$, the operator T is completely continuous in the space $C_{m,\alpha}(\bar{G})$ and

$$\frac{\partial g}{\partial \bar{z}} = f \ , \ \frac{\partial g}{\partial z} = \Pi f = - \frac{1}{\pi} \iint_G \frac{f(t)\,d\sigma_t}{(t-z)^2} \ . \tag{1.22}$$

The integral in (1.22) is understood in the sense of the prinicipal value. The operator Π is a linear bounded operator in $C_{m,\alpha}(\bar{G})$ mapping this space into itself. The operator Π can be continued up to a unitary operator in $L_2(G)$ and up to a bounded operator in any $L_p(G)$, $p > 1$. The first formula (1.22) is valid also for $f \in L_1(\bar{G})$.

Consider the operator

$$Pf = \frac{1}{\pi} \iint_G \frac{a(t) f(t)\,d\sigma_t}{t-z} \ . \tag{1.23}$$

Theorem 1.5. Let $a(z) \in L_p(\bar{G})$, $p > 2$. Then the operator (1.23) is completely continuous in the space $C(\bar{G})$, maps this space into $C_{\alpha}(\bar{G})$, $\alpha = \frac{p-2}{p}$ and

$$\| Pf \|_{C_\alpha(\bar{G})} \le M_p \| a \|_{L_p(\bar{G})} \| f \|_{C(\bar{G})} .$$

Moreover, this operator is completely continuous in the space $L_q(\bar{G})$, $\frac{1}{2} \le \frac{1}{p} + \frac{1}{q} \le 1$, too. If an integer n satisfies the condition

$$n - 1 \le \frac{2p}{p-2} \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) < n ,$$

then

$$\| P^k f \|_{L^\alpha_{\gamma_k}(\mathbb{C})} \le M_{p,q,\alpha} \| a \|_{L_p(\bar{G})} \| f \|_{L_q(\bar{G})} ,$$

$$k = 1,\dots,n ,$$

$$\| P^{n+1} f \|_{C_\beta(\mathbb{C})} \le M'_{p,q,\alpha} \| a \|_{L_p(\bar{G})} \| f \|_{L_q(\bar{G})} ,$$

where

$$\frac{1}{\gamma_k} = \frac{1}{q} + \frac{k}{p} - \frac{k}{2} + k\alpha , \quad k = 1,\dots,n ,$$

$$\beta = 1 - 2 \left(\frac{1}{q} + \frac{n+1}{p} + n\alpha\right) + n ,$$

and α is an arbitrary number satisfying the inequality

$$0 < \alpha < \frac{p-2}{2p} - \frac{1}{n} \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) .$$

The reader may find the proofs of these facts and related ones in the book I.N. Vekua [a] .

B. The Carleman-Bers-Vekua system

1. Obviously, the elliptic system

$$\bar{\partial} u + au = 0 \tag{1.24}$$

is reduced to the inhomogeneous Cauchy-Riemann equation (1.4) by taking the logarithm. Equation (1.4) entails the representation for the general solution of (1.24) in the bounded domain G

$$u(z) = \varphi(z)\exp \frac{1}{\pi} \iint_G a(t) \frac{d\sigma_t}{t-z} .$$

Here $\varphi(z)$ is an arbitrary analytic function in G. In particular, all zeros and poles of the function $u(z)$ are determined by the multiplier $\varphi(z)$. It provides a natural way to define orders of zeros and poles and to generalize the argument principle.

A more general system than (1.24) is the Carleman-Bers-Vekua (CBV) system

$$\underset{\sim}{\partial} u \equiv \bar{\partial} u + au + b\bar{u} = 0 . \tag{1.25}$$

As a rule, we assume that $a,b \in L_p(\bar{G})$, $p > 2$.

The function $u(z)$ is called a solution of the equation (1.25) in the vicinity G_0 of the point z_0 if $u \in D_{\bar{z}}(G_0)$ and the equation (1.25) is valid almost everywhere in G_0. If $u(z)$ is a solution of (1.25) in the vicinity of every point of the domain G, $u(z)$ is called a regular solution of (1.25). If $u(z)$ is a solution of (1.25) in the vicinity of every point of the domain G except some discrete set of points $G^* \subset G$, called singularities, then following I.N. Vekua [a] such a solution is called a generalized solution. Generalized and regular solutions of the inhomogeneous equation $\underset{\sim}{\partial} u = F$, $F \in L_p(\bar{G})$, $p > 2$, are defined in an analogous manner.

By Theorem 1.3 the class of generalized solutions of the Cauchy-Riemann equation $\bar{\partial} u = 0$ coincides with the class $A^*(G)$ of analytic functions in the domain G with singularities at the points of G^*; the class of regular solutions of the Cauchy-Riemann equation coincides with the class $A(G)$ of functions holomorphic in the domain G.

We denote by $\mathcal{A}^*(a,b,F,G)$ the class of generalized solutions of (1.25) such that $\bar{\partial} u = - au - b\bar{u} + F \in L_1(G)$.
It is clear that this class contains the class $\mathcal{A}(a,b,F,G)$ of regular solutions and solutions with singularities of order less than two if the coefficients of the equation at the points of G are bounded. If $a, b, F \in L_p(G)$, we write $\mathcal{A}^*_p(a,b,F,G)$ and $\mathcal{A}_p(a,b,F,G)$, respectively. The union of all classes $\mathcal{A}^*_p(a,b,F,G)$ corresponding to all a, b, F for fixed p is denoted by $\mathcal{A}^*_p(G)$ (and $\mathcal{A}_p(G)$, respectively). For $F \equiv 0$ we write $A^*_p(a,b,G)$, $A^*_p(G)$, $A_p(a,b,G)$, $A_p(G)$. All these notations are due to I.N. Vekua [a].

By Theorem 1.3 these solutions are representable in the form

$$u - Pu = \phi(z) + TF \tag{1.26}$$

where

$$Pf = - T(af + b\bar{f}) \ , \ Tf = - \frac{1}{\pi} \iint_G \frac{f(t)d\sigma_t}{t-z} \ , \tag{1.27}$$

and $\phi(z)$ is a holomorphic function in G. For $F \equiv 0$ we obtain the integral equation

$$u(z) - \frac{1}{\pi} \iint_G [a(t)\, u(t) + b(t)\, \overline{u(t)}] \frac{d\sigma_t}{t-z} = \phi(z) \tag{1.28}$$

for generalized analytic functions.

Let $a, b, F \in L_p(\bar{G})$, $p > 2$, and the function $u(z)$ in (1.26) be continuous in G. Then, by Theorem 1.1, the functions Pu and TF belong to the class $C_\alpha(\bar{G})$, $\alpha = \frac{p-2}{p}$, are analytic in the domain $\mathbb{C} - \bar{G}$ ($\mathbb{C}$ is the complex plane) and are equal to zero at infinity. It entails the representation

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{u(t)dt}{t-z} \tag{1.29}$$

<u>Theorem 1.6</u>. If $u(z)$ is a regular solution of the equation $\underset{\sim}{\partial} u = F$, $a, b, F \in L_p(\bar{G})$, $p > 2$, $u \in \mathfrak{X}_p(a,b,F,G)$, then $u(z)$ satisfies the Hölder condition, $u \in C_\alpha(\bar{G})$, $\alpha = \frac{p-2}{2}$.

For the proof see I.N. Vekua [a].

2. The following theorem is called the Bers-Vekua similarity principle.

<u>Theorem 1.7</u>. Let $u(z)$ be a generalized solution of (1.25), $u \in A_p^*(a,b,G)$, $p > 2$, and

$$g(z) = \begin{cases} a(z)+b(z) \dfrac{\overline{u(z)}}{u(z)} \ , & z \in G \setminus \{G^* \cup \{z\colon u(z) = 0\}\} \\ a(z)+b(z) \ , & z \in G^* \cup \{z\colon u(z) = 0\} \ . \end{cases} \tag{1.30}$$

Then the function

$$\varphi(z) = u(z) \exp \{- \frac{1}{\pi} \iint_G g(t) \frac{d\sigma_t}{t-z}\} \tag{1.31}$$

is analytic in the domain $G \setminus G^*$, $\varphi \in A^*(G)$.

Since $g \in L_p(G)$, $p > 2$, the right hand side of (1.31) belongs to $D_{\bar{z}}(G \setminus G^*)$ where G^* is the singularities set of the solution $u(z)$.

Then

$$\bar{\partial}\varphi = \{u(z)g(z) - au - b\bar{u}\}\exp\{-\frac{1}{\pi}\iint_G g(t)\frac{d\sigma_t}{t-z}\} = 0 .$$

almost everywhere in $G \setminus G^*$. This entails the holomorphy of φ in $G \setminus G^*$. In particular, if $u(z)$ is a regular solution, then the function $\varphi(z)$ is holomorphic in G.

Formula (1.31) involves some consequences. The most important one is the argument principle: the difference between the numbers of zeros and poles (taking their orders into account) of a generalized analytic function in the domain G is equal to

$$N_G - P_G = \frac{1}{2\pi}\Delta_{\partial G} \arg u(z) . \tag{1.32}$$

The formulae (1.28), (1.31) were obtained by N. Theodoresco [a,b], Carleman [a,b], L. Bers [a,b], I.N. Vekua [a,b].

3. Let us return to the integral equation

$$u - Pu \equiv u(z) - \frac{1}{\pi}\iint_G (a(t)u(t) + b(t)\overline{u(t)})\frac{d\sigma_t}{t-z} = g(z) \tag{1.33}$$

for the case $a,b \in L_p(\bar{G})$, $p > 2$ and show that it is solvable for any right hand side $g \in L_q(\bar{G})$, $q \geq \frac{p}{p-1}$.

By Theorem 1.5 the operator Pu is completely continuous in the space $L_q(\bar{G})$, $q \geq \frac{p}{p-1}$. Therefore, it is sufficient to show that the homogeneous equation

$$u - Pu = 0$$

has no nontrivial solutions.

Let $u_0 \in L_q(\bar{G})$, $q \geq \frac{p}{p-1}$, be a solution of equation (1.33). Then $u_0 = Pu_0 = \ldots = P^n u_0$. By Theorem 1.5 there exists such an n for which $P^n u_0 \in C_\alpha(G)$. Hence u_0 is continuous in $\bar{G}$ and satisfies a Hölder condition. By Theorem 1.4 the function $u_0 = Pu_0$ belongs to the class $D_{\bar{z}}(\bar{G})$ and, consequently, is a regular solution of the equation

$$\underset{\sim}{\partial} u \equiv \bar{\partial}u + au + b\bar{u} = 0 .$$

By formula (1.32) the value $\frac{1}{2\pi}\Delta_{\partial G} \arg u_0(z)$ is non-negative and equal to the sum of the orders of zeros of the function u_0 in the domain G. On the other hand, the function

$$u_0(z) - Pu_0(z) \equiv u_0(z) - \frac{1}{\pi}\iint_G (a(t)u_0(t) + b(t)\overline{u_0(t)}) \frac{d\sigma_t}{t-z}$$

is holomorphic in the domain $\mathbb{C} \setminus \bar{G}$ and is equal to zero at infinity. This means that $\frac{1}{2\pi} \arg_{\partial G} u_0(z) \le -1$ if $u_0 \not\equiv 0$. This contradiction proves that $u_0 \equiv 0$.

Therefore, any generalized analytic function in the domain G having poles which orders ≤ 1 is a solution of the integral equation

$$u - Pu \equiv u(z) - \frac{1}{\pi}\iint_G (a(t)u(t) + b(t)\overline{u(t)}) \frac{d\sigma_t}{t-z} = \phi(z) \qquad (1.34)$$

where the analytic function $\phi(z)$ and $u(z)$ have the same poles. Conversely, if $\phi(z)$ is an analytic function in G continuous in $\bar{G}$ up to poles of first order, then $\phi \in L_q(\bar{G})$, $\frac{p}{p-1} \le q < 2$. In this case (1.33) has a solution $u \in L_q(\bar{G})$ being a generalized analytic function in G with poles determined by the function $\phi(z)$.

4. Theorem 1.8. (Poincaré lemma). Let $a, b, F \in L_p(\bar{G})$, $p > 2$. Then the equation $\underset{\sim}{\partial} u = F$ is solvable in any space $L_q(\bar{G})$, $q > \frac{p}{p-1}$.

Consider the equation

$$u - Pu = TF \qquad (1.35)$$

The function $TF \in C_\alpha(G)$, $\alpha = \frac{p-2}{p}$ by Theorem 1.1. As it was shown above, the equation $u - Pu = 0$ has no non-trivial solutions in $C_\alpha(\bar{G})$. It entails the unique solvability of equation (1.35). It is clear, that the solution of (1.35) is a function of the class $\mathfrak{A}(a,b,F,G)$ and hence is a regular solution of the equation $\underset{\sim}{\partial} u = F$.

§ 2. The Carleman-Bers-Vekua System on Riemann surfaces

A. Riemann surfaces

Let M be a closed Riemann surface of genus g. As it is known, a Riemann surface is a topological Hausdorff space with a complex structure. The complex structure is determined by the set of simply-connected coordinate neighborhoods U such that to any point $p \in M$ there belongs at least one coordinate neighborhood. In any coordinate neighborhood U one defines the local coordinate $z(p)$, $p \in U$, mapping U into the unit disk $|z| < 1$ of the complex z-plane. If $U \cap U'$ is not empty, the relations between the corresponding local coordinates z and z' are analytic in this set and $z = z(z')$, $z' = z'(z)$ are conformal mappings.

For example, the equation

$$w^2 = (z-z_1)(z-z_2)(z-z_3)(z-z_4) \tag{2.1}$$

determines a two-sheeted surface over the z-plane. It may be constructed by attaching two copies of z-planes cut along lines connecting the points z_1, z_2 and z_3, z_4. As it is seen from Figure 1, this surface is topologically equivalent to a torus.

A compact (closed) Riemann surface is homeomorphic to a sphere with g handles. For $g = 0$ we have a sphere, and for $g = 1$ a torus. A typical property of all surfaces for $g > 0$ is the existence of cyclic sections, i.e. closed curves not separating the surface (see figure 2). For any handle there exist two kinds of such oriented sections (for a torus a parallel and a meridian one).

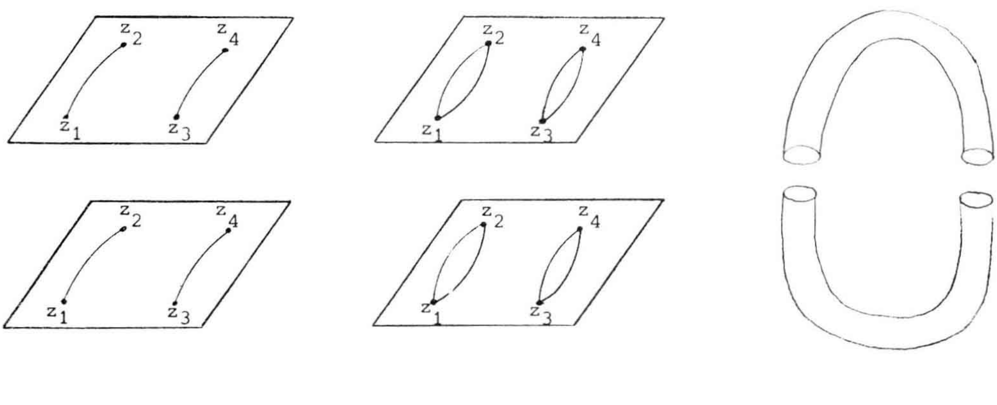

Figure 1