H. Freudenthal

Mathematics as an Educational Task

MATHEMATICS AS AN EDUCATIONAL TASK

by

HANS FREUDENTHAL

D. REIDEL PUBLISHING COMPANY / DORDRECHT-HOLLAND

Library of Congress Catalog Card Number 72-77874

ISBN 90 277 0235 7

作为教育任务的数学

本书作者是荷兰有名的数学家，先从事纯粹数学的研究，近来从事应用数学(包括实际应用和数学教学)。本书反映了作者在数学研究领域内的活动和数学教育的某些思想，如他主张要着重对中小学学生的教学，要把某些近代数学思想放到中学的数学教学中去，书中也涉及大学的某些数学教育问题。

由于数学教育的改革和数学的意义是当前广大科技和教学人员普遍关心的问题，虽然本书在某种程度上反映了资产阶级的教育观点，但仍可供有关同志分析批判地使用和参考。

本书前10章是对数学教育的一般论述(其中第2章介绍近代数学方法)，第11—14章详细地介绍了数的概念的发展，第15—19章介绍集合、函数、用变换和群的观点介绍几何、分析、概率统计、逻辑等内容，在分析中有许多物理方面的例子。

目次如下：①数学的传统，②今日数学，③传统和教育，④数学教学的用途和目的，⑤苏格拉底的方法，⑥再创造，⑦用数学化的方法构造一个领域，⑧数学的严格性，⑨教学法；⑩数学教师，⑪数的概念——客观地接近的方法，⑫数的概念的发展——从直观方法到算法化和理论化，⑬数的概念的发展——代数方法，⑭数的概念的发展——从代数原理到代数的大范围构造，⑮集合与函数，⑯几何的情况，⑰分析学，⑱概率统计，⑲逻辑。附录① Piaget 学派在数学概念上的论述，②作者在数学教学上的论文目录。

Printed in The Netherlands by D. Reidel, Dordrecht

PREFACE

Like preludes, prefaces are usually composed last. Putting them in the front of the book is a feeble reflection of what, in the style of mathematics treatises and textbooks, I usually call the didactical inversion: to be fit to print, the way to the result should be the inverse of the order in which it was found; in particular the key definitions, which were the finishing touch to the structure, are put at the front. For many years I have contrasted the didactical inversion with the thought-experiment. It is true that you should not communicate your mathematics to other people in the way it occurred to you, but rather as it could have occurred to you if you had known then what you know now, and as it would occur to the student if his learning process is being guided. This in fact is the gist of the lesson Socrates taught Meno's slave. The thought-experiment tries to find out how a student could re-invent what he is expected to learn.

I said about the preface that it is a feeble reflection of the didactical inversion. Indeed, it is not a constituent part of the book. It can even be torn out. Yet it is useful. Firstly, to the reviewer who then need not read the whole work, and secondly to the author himself, who like the composer gets an opportunity to review the *Leitmotivs* of the book. Though I just did it with one of them, I would prefer not to continue this way. I would rather try to do justice to some of the features I seem to have neglected a little.

The present book is not a methodology of mathematics in the sense that I will systematically show how some teaching matter should be taught; it is not even a systematic analysis of subject matter. I hardly ever refer to well-organized classroom experiments evaluated by statistical methods, nor do I cite experimental results of developmental psychology or the psychology of learning. Maybe the most striking feature is that this book contains few quotations. I will try to justify all these features.

First, to take the psychological literature. To be honest I should say that I feel there is no need to embellish low-key education using highbrow psychology, in particular if the cited literature is far removed from educational preoccupations. If others prefer this procedure, then indeed, I feel the need to oppose it. Misusing Piaget's name has become quite a habit in didactical literature. This led me to discuss in passing, and finally in a more connected form in the Appendix, what Piaget's investigations could mean for mathematical education.

Perhaps in mathematics instruction one expects more from the psychology of learning, and in a more technical sense this expectation may certainly come true in detailed investigations. I found a lot of interesting and even exciting things in the psychology of learning though hardly anything I was looking for. When in an excellent modern book* I tried to find out what I should understand under learning and how I should subdivide it, I felt myself very far from what I had experienced myself and with others as *mathematical* learning. A feeling of loneliness seized me: is mathematics really so different? I wish that someone who profoundly understands both mathematics and psychology would show us the bridge.

Except for some general ideas I did not take my empirical material from psychology. My most direct sources are textbooks, didactical designs, actual lessons, as well as observations with individual children; indirectly my main sources were talks and discussions with teachers. With respect to the second kind of source one can find an acknowledgement of some of the influences at the end of this Preface. On the other hand I avoided all citations with respect to textbooks, designs and lessons wherever it was feasible. I believe I had a compelling reason to do so for this material was frequently subjected to criticism, which in fact was often negative. The material could be sharply divided into serious work and trash. Citations in footnotes would have meant tarring everything with the same brush. This I would hate to do. At the same time, it would have been too much an honour for trash to be quoted along with the serious literature. Therefore, I have made explicit quotations only in a few quite specific cases.

For a few other reasons I did not mention mathematical-didactical investigations. The main reason is that except for some generalities I did not use their results because I could not. I will explain why.

* R. M. Gagné, *The Conditions of Learning*, London 1965.

The first kind of investigations I have in mind are of the kind that intend to show that some particular subject matter is teachable. The author submits the subject matter to us; he tells us where and when it was tested, and gives, or does not give, statistical data about its success. Mostly there is no additional data relating to the teaching methods, and this makes the report worthless, because without any further experimentation it may be taken for granted that with the appropriate methods all you want to test can be crammed into the children's heads. Quite recently I saw a course for individual instruction (an excellent one, I should add), where, indoctrinated with a wrong recipe, the children obediently proved the same nonsense for quite a number of years and never protested – and this therefore proves that this subject matter was "teachable"!

There is still, however, a more serious reason why I do not believe in such investigations. At most they prove that the subject matter is learnable, not that it is teachable. It is not true that this means the same thing. That a subject matter is teachable by a few does not imply that a sufficiently large number of teachers can teach it. If it is mathematically wrong, or didactically mistaken or worthless, quite a few teachers may simply refuse to teach it or will do it with so much distaste that it ceases to be teachable. Further, some subjects can be so uncommon that they can only be taught if it is told in all details how it should be taught, but such details are usually lacking. I mean by this, indications on the form of instruction which is best adapted to the subject matter, rather than on didactic details. This is a point which has perhaps been a bit neglected in the present book. If we design teaching material and methods, we should not only weigh up what can be learned and is worth learning, we should also be concerned about what kind of subject matter the teacher can learn to teach, or rather what we can teach our teachers to teach their pupils – if I look back on my own activities and on this book, I am not prone to estimate my own capabilities too high in this regard.

I am going to continue discussing the kind of investigations I was not able to use fruitfully. A second kind are those where with respect to a particular subject matter two teaching methods or subject arrangements are compared, say, with the result that the author has inferred with a probabilistic certainty of 98% that one method is not as bad as the other. About thirty years ago was the first time that I saw such an investigation;

not in mathematics but in geography. The investigation was above reproach; the only thing that surprised me was that it was still the same geography which had been the most boring chapter of my own school career. Since then I have seen a lot of similar investigations, and as far as the authors described their teaching methods, I often could not believe that what I was reading was still possible today.

Maybe these were exceptions, but however technically perfect such investigations may be, they cannot answer the preliminary educational questions what, for which purpose, and to whom is a subject matter being taught. My criticism is aimed at the spirit behind such research. Embellishing it with a statistical analysis does not mean that the rigour of natural science has been transferred to educational research. The only thing that reminds one of (bad) natural science is the pretension that the seventh digit after the decimal point is correct while everything left of the point is wrong. Rather than from such experimental investigations, I learned a lot from my own and from reported classroom experiences, from textbooks and manuals, whether I liked them or not, and from honest analysis of subject matter and learning behaviour, as performed by experienced teachers.

True educational activity means tracing the right path to education, guided by one's own honest conviction. Educational science should, first of all, be the rational justification of this honest conviction. You may call it philosophy. But whatever it is called, it cannot be missed. Investigations on details cannot replace it, on the contrary, they can flourish only in the soil of a healthy educational philosophy.

In spite of all the detailed investigations in this work this book is above all a philosophy of mathematical education. I am not the first to have written such a book. The least one should have learned when studying his predecessors is that one has come to terms with their ideas. The scientific character of a book like the present one is not measured by the number of footnotes but by the thoroughness of this preliminary discussion.

I have often lectured and written about teaching. This book does not contain any essentially new material compared with my earlier papers; in a few places I have even reproduced texts that have already been published before. Here I have taken the opportunity of rearranging my old ideas. As a mathematician I did not feel that it was an easy job.

The problem was not the dialectic instead of the deductive style, and the local organization of the subject matter was not a problem either. But the *global* organization was the sore point. I could not use the formal organization of a mathematics course or treatise where the author says, or writes things like "because of theorem... (cp. p. ...), applied under the condition of corollary... (p. ...), it appears that the definitions of ... on p. ... and on p. ... are equivalent." I could not use this method nor could I invent another form of organization. Thus the present book is, from the viewpoint of a mathematician, badly organized. Numerous repetitions were unavoidable.

Though I cannot cite in detail how much I have learned from others, I am fully conscious about its importance and acknowledge my debt. The first suggestions to occupy myself theoretically with education came from my wife during the course of common educational studies. Among pedagogical psychologists I believe that I was most strongly influenced by O. Decroly. My educational interpretation of mathematics betrays the influence of L. E. J. Brouwer's view on mathematics (though not on education). From 1945 to 1963 I learned much of principal importance as well as many didactical details in the mathematical working group of the Dutch section of the New Education Fellowship; among its members to whom I owe so much, I need only mention the names of P. M. van Hiele and his late wife Dieke Geldof. More recently, thanks to activities in the international educational field, my circle of friends has been enlarged; I am grateful for all I learned at international meetings, and in particular my thanks extend to Emma Castelnuovo, Zofia Krygovska, and W. Servais, and more recently, to A. Revuz. My book is dedicated to all who are committed to mathematical instruction.

Utrecht, 27 December 1970

LIST OF SYMBOLS

$\vee$ or, $\wedge$ and, $\bigvee$ there is a ..., $\bigwedge$ for all ...,

$\rightarrow$ if ... then, $\neg$ non-,

$\cup$, $\bigcup$ union, $\cap$, $\bigcap$ intersection,

$\in$ adherence, $\subset$ inclusion, ...\\... without,

$\bigcirc$ void set,

$\#$ number of ...,

$\ulcorner a, b \urcorner$ ordered pair a, b,

$\ulcorner A, B \urcorner$ set of the $\ulcorner a, b \urcorner$ with $a \in A$, $b \in B$,

$A \rightsquigarrow B$ set of the mappings of A into B,

$\{x|...\}$ set of the x with the property ...,

$\curlyvee_x$..., ... as a function of x,

N set of the natural numbers,

Z set of the integers,

Q set of the rational numbers,

R set of the real numbers,

C set of the complex numbers,

with superscript $+$, ... positive ...

CONTENTS

CHAPTER I

THE MATHEMATICAL TRADITION

Somebody once made the funny and effective remark mathematics was she

> who first is sneaking with a low frame, but soon raises her head to the heavens and walks on the earth *Iliad* IV, 442–3, speaking of Eris

because it starts with the point and the line, but its investigations comprise heaven, earth, and universe.

Hero, *Definitiones*

Non igitur lector lacrimes? decepit utrosque
maxima mendacis fama mathematici. (*CIL* VI, 27140)
Not for this reason weep – look to my parents, misled by the great renown of the mendacious mathematician.

From the epitaph of four year-old Telephus on the via Appia

If the army of the enemy sets out 6 days ago and is marching $3\frac{1}{2}$ miles* daily, and ours starts today, how many miles should it march a day to catch up in a week?

From a German arithmetic book of 1799

Nobody knows what man invented first, writing or arithmetic. The alphabet is two millennia older than our present Indo-Arabian numerical system but this in itself proves nothing. Mathematics is much older than those numerals. The first exercises in writing and arithmetic are closely connected with each other. Whether, how, how much and how long before this time people counted orally or with counters, nobody can tell. It is a striking fact that the numerals up to 10 and that for 100 belong to the common stock of the Indo-European linguistic family. They were invented long before writing.

No matter how they developed, by the end of the third millennium B.C. well-groomed elementary arithmetic and algebra existed in Babylonia. It is not our formal algebra of x and y. The unknowns are indicated by the terms "length" and "width" (of a rectangle). It is said that science in Babylonia was the business of priests. But this term is a misleading one. What they called priests were in fact the intellectuals of that era, the clerks, the teachers, the librarians, the star-gazers, the

* A German mile equals about 5 statute miles.

augurs and haruspices, the soothsayers, the temple and palace architects, and the sorcerers. At the cradle of mathematics stood the calculator, the surveyor, the merchant, the money-changer, the banker, the book-keeper, the executor, the publican, and the builder of bridges, roads and cities. Yet their wants were quickly satisfied. The problems students had been solving in the temple schools of Babylonia for two millennia were not quite as practical. Teachers let them cover a road 100 km long and one mm wide with asphalt and then compute how many day's wages it cost. Or they posed the problem of inheritance of 65 gold coins, divided among five brothers such that the younger brother gets three coins fewer than his immediate older brother. Every generation solves these immortal problems anew – the stone that weighs one pound more than half its own weight, or the lance that towers one cubit above the wall against which it is vertically placed and which recedes three cubits if it leans against the wall without exceeding it.

Indeed, this is what they learned – useful multiplications and divisions with tables and counters. But with what purpose did they do so? To solve useless linear and quadratic equations? One wonders if they ever complained? And if they did, what did their fathers or their teachers answer?

Did they reply that students had been learning mathematics as early as the flood, that mathematics was a whetstone of wit, or simply that other topics were even more useless? For instance Sumerian, which was still being taught when it had already been extinct for two millennia, or Accadian, and the cuneiform script, when they spoke Aramaic in the streets of Babylon and when the alphabet had been in existence for the last thousand years. Or did their teachers answer – wait a few months, and next year you will be told how this mathematics can be applied to compute the calendar, the feasts, and the course of the sun, the moon and the stars?

Astronomy was the next science of mankind, and mathematical astronomy was two millenia younger than mathematics itself. This was indeed a practical science. You cannot conjure stars and planets out of a void just like cooking up mathematical problems. What was the use of astronomy? To foretell the calendar, feasts, eclipses, wars, pestilences, whirlwinds, storms, inundations, the fortune of nations, and finally even of individuals. Was it not a useful science, a useful application of mathematics that would live on for two millennia? In fact, even in these celestial applications, there was no use of quadratic equations. To apply these, one had to rely on problems like this one: "Length and width, I multiplied length and width and got the area; I added the excess of length over width to the area, 183; I added length and width, 27; asked for: length, width, area." Thousands of such problems have been preserved on clay tablets, though very little remains of the theoretical literature, the "teaching texts" with the rules for solving such problems.

Even less has been conserved of Egyptian mathematics, which was not entrusted to clay tablets, but to much less durable papyrus. But the principle is still the same: it was a mathematics that quickly and to a great extent surpassed practical needs. We can only too well understand why the calculator and the surveyor were fascinated by the figures and shapes they were familiar with, that they liked to play with these objects, to unravel their secrets, and to fathom their mysteries. Most books tell us that up till the time of the Greeks, mathematics was a collection of basic applications, but this is simply not true.

Certainly, Greek mathematics was different, and if the few sources that have survived are trustworthy, it was different from the very beginning. Some time in the sixth century B.C., the Greeks must have learned Babylonian mathematics and astronomy. In what is traditionally known about Thales, Babylonian influence is easily recognizable and Babylonian mathematics accounts for much that is told about Pythagoras and his School. Who does not know the so-called Pythagorean theorem? This theorem was known by the Babylonians some two millennia earlier than the Greeks. Was Pythagoras perhaps the first to prove it?* No, a theorem like this which is not obvious by mere sight, can only be discovered by proving it; it cannot be found empirically by measuring the sides of triangles. Yet most books will tell you that proving theorems was a Greek rather than a Babylonian invention.

What the Greeks actually did was to make demonstration a principle in mathematics. In Greek mathematics is outlined what is today called a deductive system. This indeed possibly started with Thales. He is said to have proved theorems, and a closer look at these propositions reveals they are not the kind of the Pythagorean theorem but like that

* Contrary to modern tradition, there is no indication that the ancient Egyptians ever knew it.

of the equality of angles in an isosceles triangle, that is, theorems which are obvious by sight alone. When people start proving such propositions, they betray they have discovered a new game, namely demonstrating for demonstrating's sake. From the fact that they were able to prove such propositions, we may conclude that they had constructed a system in which demonstrating is a meaningful activity. If such a system and such a method of demonstrating ever existed in Babylonia, all traces of it have vanished. Aristotle expounds what a deductive system is more clearly than has ever been done up to modern times. Every true science, according to Aristotle, starts with "archai", principles, on which it rests by its very nature and from which it can be derived. Euclid's Elements start with Definitions, Postulates and Axioms; other authors use other terms, but this custom of starting geometry with such principles was at least one century older than Euclid's Elements. Probably Hippocrates of Chios, the first author of Elements, already knew it. We do not know the origin of this custom, whether it sprang from philosophy or from the discussion techniques of public meetings. One can imagine that such a stock of principles was a means of fighting chicanery and litigation.

Euclid did not explicitly account for all the axioms he used but blaming Euclid for this incompleteness is too modern a stand-point. A science rests on principles but nobody asks you to enumerate all of them; how far to go is open to dicussion.

There are parts of Euclid that look like modern mathematics, for example, the theory of proportions and similarity in the 5th and 6th books. It is ascribed to Eudoxus; it plays the part which we today allot to the theory of real numbers. There are, on the other hand, parts which have an extremely weak deductive structure. Euclid's work was essentially a compilation. Nevertheless, for well on twenty centuries it excited admiration and invited imitation. This admiration was justified, but imitations were not usually very successful. Of course, people like Archimedes and Christiaan Huygens have been as great axiomaticians as Eudoxus, but the axiomatic efforts of Spinoza's philosophy more geometrico, Leibniz' example of axiomatic jurisprudence and politicology, Whiston's axiomatic cosmology, and whatever else produced in this field, all were not truly convincing. What axiomatics means and how its axioms should be formulated was not shown until the end of the 19th century, in Pasch's work – he taught it to the Italian geometers – and Hilbert's.

Deductivity and the germ of axiomatics are in our view the most striking, and in fact the most modern, feature of Greek mathematics. Another great feat of Greek mathematics was the discovery of the irrational, the incommensurability of diagonal and side of a square. There are few things that look more obvious than that every ratio of magnitudes can be expressed by natural numbers. The discovery that this was not true should have caused a crisis in the foundations of mathematics according to modern historians, but this is probably too modern a view. It is true that it did not fit into the Pythagorean doctrine that all was number, but the mathematicians among the Pythagoreans tried to find a way out. A fresh definition of ratio was required, though not by natural numbers. It was first done by infinite approximations, which finally were again eliminated. The definitive solution in antiquity was something like Dedekind's cuts. It is presented in the 5th and 6th books of the Elements, along with the antique version of epsilontics. Greek doctrine took a further step: not only infinite processes, but also Babylonian algebra were eliminated. Since numbers did not suffice to explain geometrical ratios, they were banished from geometry; real numbers were unknown, and rational numbers forbidden – in exact science that is. Merchants and craftsmen continued to use fractions. To the mathematician, as to Pythagoras, number, that is, the natural number was sacrosanct. Plato reacted with irritation to attempts "to divide the unit".

Was algebra thrown away? No, not completely, for a surrogate was invented, a geometrical algebra, a system of geometrical mummery of algebraic operations, of linear and quadratic equations, and of solving procedures. This system was expounded in Euclid's 2nd book and, was applied in the 10th in particular, in the classification of irrationalities – a paragon of unreadable mathematics.

Geometrical algebra, this impractical product of methodical dogmatism and fanatical rigourism was the disease which killed Greek mathematics. As long as the heuristic methods of algebra and infinitesimals were still taught orally alongside the official Euclidean-Archimedean rigorous mathematics, students could learn to work within the official straitjacket. As soon as this tradition was interrupted, all was lost. As late as the 3rd century A.D. the Babylonian tradition seems to have survived – this is proved by the existence of Diophantus, who was a genuine algebraist – but this, then, was the last flare. Algebra was re-invented in the Arabian world, and both the Indians and the Christian Middle Ages contributed to its revival though Greek rigour still dazzled the heirs of Greek culture. The first to cut free from the Greek tradition was Descartes, the challenger of all tradition. He put the cart before the horse: rather than geometrizing algebra, he algebraized geometry. The result was what in school and university instruction used to be called analytic geometry. Meanwhile, limit procedures and infinitesimal methods had come into vogue and finally led to the invention of calculus (Newton's fluxions and Leibniz's differential and integral calculus). Nobody actually realized what scruples had led the Greeks to reject algebra; Eudoxus' epsilontics was not understood, or if it was, it was rejected. Euclidean-Archimedean rigour was still admired, but there were scarcely any who really understood it, and after Christiaan Huygens there was nobody left to imitate it. Not until rigour was recaptured in the 19th century did people understand the essence of Greek mathematics. Maybe this course of events was a historic necessity: Eudoxus' strait-jacket that choked Greek mathematics, the non-mathematical millennium, the liberation when the good was cast away with the bad, the laborious reconstruction of rigour (which lasted longer than in antiquity) and finally the rediscovery of the Greeks who long ago already knew so much – perhaps each link was historically indispensable.

So much about the tradition of mathematical rigour. Once more I must warn against exaggerated ideas on antique rigour. Elementary geometry in Euclid in particular shows gaps, and even sham arguments. On the other hand, the modern reader is struck by the care that was bestowed on the theory of parallels. The postulate on parallels, such as it is found in Euclid, was in antiquity the final solution of a problem that must have preoccupied Greek mathmaticians for a long period before Euclid. From rare allusions to other views on parallels one can guess that the Greeks knew more about it than what has been handed down in the Elements, and that they were nearer to the historically still remote non-Euclidean geometrics than straight historical data would seem to allow. Yet again, as in the case of mathematical rigour, the content of tradition in foundations of geometry was fixed for two millennia by Euclid's Elements. The same is true of the geometrical method, the well-known auxiliary lines by which figures are parcelled out into sequences of congruent triangles which permit a systematic walk from one to the next to forge a chain of congruence relations between two magnitudes which are to be proved equal – a methodical madness. Take, for instance, such a classical problem of school mathematics as proving that in a cube, given a vertex A, the plane through three vertices that are joined with A by edges, is orthogonal to the space diagonal through A. How many congruent triangles are needed to prove it, whereas mere sight shows that the rotation through 120° with a space diagonal as an axis, leaves the cube and the aforesaid plane invariant, which proves the assertion immediately! A decennium or two ago such a proof would have been considered improper. Fortunately today mappings like reflections, translations, and rotations are *dernier cri* in school instruction. In creative geometry, mappings emerged in the 19th century; they are a principle of modern geometry, but the Euclidean tradition of congruent triangles was still in this century so coercive that even as great an authority as Felix Klein did not succeed in introducing mappings into German school instruction. Up to the last "Elements" before Euclid, mappings seem to have been an admissible argument; though some relics have survived in Euclid's Elements, it is a fact that Euclid weeded out geometrical mappings and that this revision decided their fate up to the 19th century. Why were mappings outlawed? Probably because their kinematic undertone was out of tune with the lofty static character of geometry; geometry's detachment from the material world did not tally with the variability which is characteristic of motion – such philosophical dogmas, which are still heard in more modern times, may have been the afterthought when mappings were rejected. So strong was the Greek tradition that even the kinematically-motivated modern concept of function did not modify geometrical habits.

Pythagoras, according to ancient tale, raised geometry from an artisan's business to a liberal art, that is, to an occupation of a free man who does not soil his hands. Together with Arithmetic, Music and Astronomy, Geometry belongs to the four 'non-trivial' arts of the medieval quadrivium. All of them were ascribed to Pythagoras; it is a fact that they were at least taught by his first disciples. The term "mathematics" sprang from that circle. Among Pythagoras' adepts there was a group that called themselves mathematicians, since they cultivated the four "mathemata", that is geometry, arithmetic, musical theory and astronomy.