Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann
Subseries: Mathematisches Institut der Universität und
Max-Planck-Institut für Mathematik, Bonn – vol. 13
Adviser: F. Hirzebruch

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Periods of Hecke Characters

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): Primary: 10D25, 14A20, 14K22, 12C20
Secondary: 14C99, 18F99, 14G10, 12C25, 33A15, 14K20

ISBN 3-540-18915-7 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18915-7 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

A Rosita

qui a su débloquer la rédaction
de ce travail
janvier 1986

Налево беру и направо,
И даже, без чувства вины,
Немного у жизни лукавой
И все – у ночной тишины.

Анна Ахматова

INTRODUCTION

In two papers – n^o 12 and 14 of [He], published in 1918 and 1920 – E. Hecke introduced what he called "Größencharaktere" of algebraic number fields, with a view to extending the theory of L-functions and their applications in analytic number theory. In the early 1950's the arithmetic and geometric significance of those of Hecke's characters that take algebraic values began to appear in two different, if overlapping, lines of thought. (Both of these had been anticipated in special cases by Eisenstein exactly one hundred years earlier; but none of the mathematicians working on them in the fifties seems to have been aware of their precursor at the time.) First Weil, testing a conjecture of Hasse, investigated algebraic curves over $\mathbb{Q}$ with the property that the number of $\mathbb{F}_p$ rational points on their reductions modulo p can be computed in terms of exponential sums. This led him to a study of "Jacobi sums as 'Größencharaktere' ". Secondly Deuring, developing one aspect of Weil's examples, proved that the (Hasse-Weil) L-function of an elliptic curve with complex multiplication is a (product of) Hecke L-function(s). This was then quickly generalized to higher dimensional *CM* abelian varieties by Shimura and Taniyama, with Weil providing clarification, for instance, on the Hecke characters employed in the theory.

Both approaches cover only very limited classes of algebraic Hecke characters. Jacobi sum characters were confined to cyclotomic (today: abelian) fields, and in general, not every algebraic Hecke character of such a field is given by Jacobi sums. – The product of several Hecke characters each one of which is attached to a *CM* abelian variety does no longer occur in the L-function of an abelian variety.

This last difficulty disappears in a theory of motives, as proposed by Grothendieck. There one associates with every (smooth projective) algebraic variety, in some sense, its "universal cohomology" which is an object of what Grothendieck termed a Tannakian category, and may therefore be viewed as a representation of some proalgebraic group. The product of Hecke characters then corresponds essentially to the tensor product of representations. Until the mid seventies such a category of motives existed only conjecturally; the morphisms were to be defined by the cohomology classes of algebraic correspondences, and conjectures on the existence of sufficiently many algebraic cycles had to be used to show that the construction actually yielded a Tannakian category – see [Sa]. Granting this, the semi-simplicity of the Galois action on l-adic cohomology and Tate's conjectures, one could show that a motive defined over a number field is determined up to isomorphism by its ("Hasse-Weil") L-function (defined using the étale cohomology of the motive). Consequently, two motives (which may be constructed from different varieties, but are) attached to the same Hecke character would have to be isomorphic, and in particular, would have the same periods (defined by "integrating" de Rham cohomology classes "against" Betti cohomology of the motive.)

This uniqueness principle is at the center of our work. We peruse a variety of consequences of it that can be proven, either because an analogous uniqueness principle is available in a slightly different framework – see next paragraph – or because of the special situation considered – this is the case in Chapter V. Applications include a refined version of the so called formula of Chowla and Selberg, deduced from the comparison of the motive of a basic Jacobi sum Hecke character of an imaginary quadratic field K to elliptic curves with complex multiplication by K – see Chapter III; refinements of Shimura's monomial period relations; generalizations of the formula of Chowla and Selberg to arbitrary abelian number fields – Chapter IV; and the study of motives for the theta series of Hecke characters of imaginary quadratic fields - Chapter V.

That we can actually prove theorems, not merely do an exegesis of conjectures, hinges on two insights by P. Deligne. First, he saw that one could actually construct a theory of motives by weakening the requirement on the morphisms; they no longer have to be algebraic but only "absolute Hodge" correspondences – see Chapter I, Section 2. (This idea may in fact go back to Grothendieck: see the first consequence of the Hodge conjecture discussed in §4 of [Gro].) Henceforth, when we speak of "motives", we refer to this existing theory. Second, Deligne was able to show that, on an abelian variety over $\bar{\mathbb{Q}}$, every Hodge cycle is an absolute Hodge cycle. This consequence of the Hodge conjecture provides enough absolute Hodge cycles to prove the uniqueness principle for motives of algebraic Hecke characters, within the category of motives generated by abelian varieties – see Chapter I, Theorem 5.1.

In fact, for every algebraic Hecke character of a number field K, there exists a unique motive in the category of motives over K generated by abelian varieties with potential complex multiplication. Deligne has shown around 1980 that this category is equivalent to the representations of (a subgroup of) the Taniyama group, a group scheme which had been introduced by Langlands. This structure theorem also links the motivic interpretation of Hecke characters to that proposed by Serre in [Sℓ] more than ten years earlier. It was also the starting point for G. Anderson's comprehensive motivic theory of Gauss and Jacobi sums, and their relations to representations of the Taniyama group, a theory which he worked out between 1982 and 1984 – see [A1] and [A2]. In Anderson's formalism, the basic observation that Fermat hypersurfaces provide motives for Jacobi sum Hecke characters of cyclotomic fields is extended to a class of characters of abelian number fields which is likely to include all sensible candidates of Hecke characters of "Jacobi sum type". We make essential use of Anderson's theory when dealing with Jacobi sum Hecke characters.

Thus, I really "take on the left and on the right" very substantial results obtained by others, and numerous little chats with many people have found their way into the "silent hours of the night" during which these pages were written.

* * *

It was my intention, in writing up the paper, to also provide a viable introduction to the background theories. More precisely, the reader should get an idea of what they are like, without however being offered complete proofs. I hope there will be readers to whom my blend of explanations and quotes appeals, and is actually helpful.

CHAPTER 0 should be completely readable for anyone with some very basic knowledge of algebraic number theory. It covers the elementary (as opposed to geometric) theory of algebraic Hecke characters, including their interpretation via Serre's groups $S_{\mathfrak{m}}$, and the definition and basic properties of Jacobi sum Hecke characters according to G. Anderson. (The Jacobi sum characters of imaginary quadratic fields are largely treated without reference to Anderson, by way of a fundamental example which is used in Chapter III.)

CHAPTER I falls into five parts.

I §1 presents the Shimura-Taniyama theory of complex multiplication of abelian varieties with a view to introducing motives for Hecke characters. The existence of the Hecke character attached to a CM abelian variety is derived using a transcendence result which implies - see [Henn] – that every semisimple abelian E-rational λ-adic representation is locally algebraic – cf. I, 1.4.

I §2 reviews the theory of motives for absolute Hodge cycles. We hope that our shortcut through this theory can serve as a reading guide for [DMOS], Chapter II, and also to the corresponding sections of [A2]. Deligne's fundamental theorem on absolute Hodge cycles of abelian varieties is only quoted from [DMOS], Chapter I, because its proof would have led us too far away from the geometric study of Hecke characters.

I §§3 - 5 cover the "naive" theory of motives for Hecke characters. In §4, a motive for every algebraic Hecke character is constructed "by hand", out of Artin motives and CM abelian varieties. Its uniqueness up to isomorphism, in the category of motives generated by abelian varieties, is derived from Deligne's theorem in §5.

I §6 treats the theory of the Taniyama group and its relation with the category of motives $\mathcal{CM}_{\mathbb{Q}}$. While in the previous sections of chapter I the reader should be able to survive with a certain knowledge of algebraic geometry, this section is deliberately sketchy. In fact, we shall make very little use of it in later chapters – except through Anderson's theory. Also, Milne is preparing a book on this subject which will also deal with Shimura varieties.

I §7 briefly reviews Anderson's theory of motives for Jacobi sum Hecke characters, and also his ulterior motives. For all the details the reader is referred to his papers.

CHAPTER II is the technical heart of this work. The formalism of the periods of motives in general and motives for Hecke characters in particular, is unfolded here. This "arithmetic linear algebra" is carried out in great generality. I am afraid this does not exactly simplify the notation and understanding of this chapter. But I do hope that this treatment of periods – which, by the way, is essentially due to Deligne – will be useful for further investigations. This chapter also contains a brief review of Deligne's rationality conjecture for special values of Hecke L-functions. This case of the conjecture is now a theorem by virtue of recent important results of Blasius [Bl] and Harder (unpublished). However, Blasius's motivic treatment of the periods c^+ is not included in my exposition because he himself applies it to questions similar to those discussed here – see his forthcoming paper [Bl′].

At the end of chapter II, after discussing the periods of Jacobi sum Hecke characters starting from the example of Fermat hypersurfaces, we deduce some relations between values of the Γ function at rational numbers which were first conjectured and proved by Deligne.

CHAPTER III is devoted to the so called formula of Chowla and Selberg. We prove a refined version of it, and show (in III §3) that it "generates" all period relations produced by Jacobi sum Hecke characters of imaginary quadratic fields. An interesting feature of the motivic treatment of this formula is that here, it is often convenient to deduce an identity of Hecke characters from an analytically accessible period relation – rather than going the other way around. – Cf. also II, 3.5.

CHAPTER IV treats Shimura's relations between periods of CM abelian varieties and generalizations of the Chowla-Selberg formula to abelian fields. The most remarkable feature here is the serious discrepancy between the potential of the method and the scarcity of information about concrete situations to which the method applies. In the Chowla-Selberg case it is often possible to determine explicitly every single character whose periods contribute to the formula. But over an arbitrary abelian field such explicit identities are usually not available, and so – in spite of the inherent precision of the method – one is led to weaken the period relations in order to get sensible statements.

Compared to the preceding chapters, CHAPTER V is really written "in shorthand". It starts by reviewing very briefly U. Jannsen's recent construction of an honest regard (absolute Hodge cycle) motive for every newform f on $\Gamma_1(N) \subset SL_2(\mathbb{Z})$ of weight ≥ 2. Then we proceed to show that this motive "lies in" $\mathcal{CM}_{\mathbb{Q}}$ if f has complex multiplication. This has to be done by hand, using Deligne's conjecture for the critical values of these modular forms.

* * *

It is a pleasure to acknowledge the hospitality of the Max-Planck-Institut für Mathematik at Bonn, where I stayed from October 1983 through January 1985. About half of this work was written there, not little influenced by Harder's interest in these questions, and his willingness to let me organize his seminar in the winter 1984 - 85 on motives for absolute Hodge cycles. Most of the suggestions explicitly acknowledged in the text I obtained through my stay in Bonn. – For the excellent typing my hearty thanks go to K. Deutler at Bonn, C. Gieseking at Göttingen.

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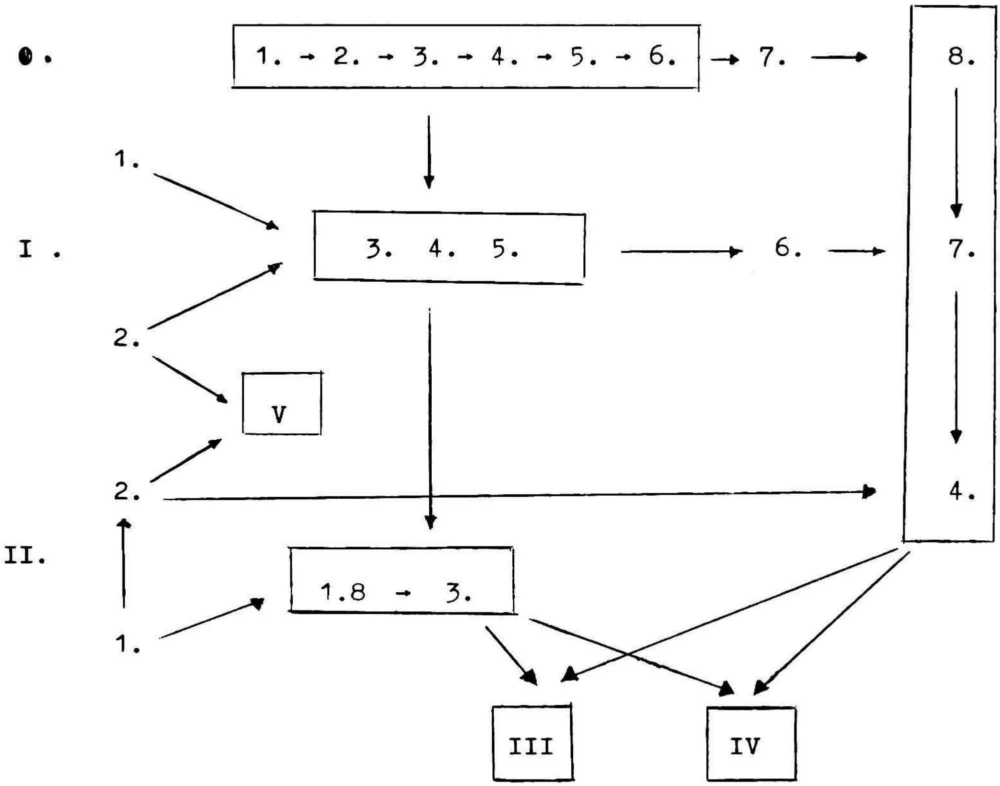

Here as well as in the internal references throughout the text, roman numerals denote chapters - chapter zero being represented by Ø - , and an expression of the form n.m.l , n.m , or n. refers to the corresponding formula, theorem, paragraph or section within the given chapter.

CONTENTS

CHAPTER ZERO:
Algebraic Hecke Characters

In this chapter we review the elementary theory of algebraic Hecke characters and fix some basic notation.

1. Definition

Let K and E be two number fields, i.e., finite extensions of $\mathbb{Q}$. Let $\mathfrak{f}$ be a non-zero integral ideal of K, and $T = \sum n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(K,\bar{E})]$ a $\mathbb{Z}$-linear combination of embeddings of K into a fixed algebraic closure $\bar{E}$ of E.

Definition: [cf. [SGA $4\frac{1}{2}$], Sommes trig. § 5]: An algebraic Hecke character χ of K with values in E, of infinity-type T and conductor dividing $\mathfrak{f}$, is a group homomorphism

$$\chi: I_{\mathfrak{f}} \to E^*$$

from the group $I_{\mathfrak{f}}$ of ideals of K prime to $\mathfrak{f}$ to the multiplicative group of E, such that, for any ideal $(\alpha) \in I_{\mathfrak{f}}$ generated by an $\alpha \in K^*$ with $\alpha \equiv 1 (\mathrm{mod}\ \mathfrak{f})$, and α totally positive (i.e., $\alpha^\rho > 0$ for all real embeddings $\rho: K \hookrightarrow \mathbb{R}$, symbolically: $\alpha >> 0$), one has

$$\chi((\alpha)) = \alpha^T = \prod_\sigma (\alpha^\sigma)^{n_\sigma} .$$

It is understood that, if $\mathfrak{f} | \mathfrak{f}'$, characters of conductor dividing $\mathfrak{f}$ are identified with the corresponding characters of conductor dividing $\mathfrak{f}'$ obtained by restricting to $I_{\mathfrak{f}'} \subseteq I_{\mathfrak{f}}$. The smallest $\mathfrak{f}$ (in the sense of divisibility) such that χ extends to a character of conductor dividing $\mathfrak{f}$ is called the conductor of

χ, and denoted f_χ. - Note that the subgroup of ideals (α) with $\alpha >> 0$ and $\alpha \equiv 1 \pmod{f}$ has finite index in I_f.

2. Algebraic homomorphisms

Recall from [SGA $4\frac{1}{2}$], Sommes trig. § 5, the various ways to view the infinity-type T of an algebraic Hecke character. In general, an <u>algebraic homomorphism</u> $t: K^* \to E^*$ is a group homomorphism such that either one of the following equivalent conditions is satisfied.

(a) For any basis $\{e_i \mid i = 1,\dots,n\}$ of K over $\mathbb{Q}$, there is a rational function $f \in E(X_1,\dots,X_n)$ such that

$$t(\sum a_i e_i) = f(a_1,\dots,a_n),$$

for all $(a_i) \in \mathbb{Q}^n$.

(b) t is induced by a homomorphism of algebraic groups over E

$$R_{K/\mathbb{Q}}\, \mathbb{G}_m \times_{\mathbb{Q}} E \to \mathbb{G}_m .$$

(c) t is induced by a homomorphism of algebraic groups over $\mathbb{Q}$

$$R_{K/\mathbb{Q}}\, \mathbb{G}_m \to R_{E/\mathbb{Q}}\, \mathbb{G}_m.$$

(d) There is $T = \Sigma\, n_\sigma\, \sigma \in \mathbf{Z}[\mathrm{Hom}(K,\overline{E})]$ such that for all $\alpha \in K^*$,

$$t(\alpha) = \alpha^T = \prod_\sigma (\alpha^\sigma)^{n_\sigma}.$$

(e) Decompose $K \otimes_{\mathbb{Q}} E = \prod_j F_j$ (finite product of fields).
There are integers m_j such that

$$t = \prod_j N_{F_j/E}^{m_j} .$$

As explained in loc. cit., the equivalence of (a) through (c) follows from elementary facts about algebraic groups, and (d), (e) are reformulations of (b) using the identification of the character group of $R_{K/\mathbb{Q}}\mathbb{G}_m$ over $\bar{E}$ with $\mathbb{Z}^{\mathrm{Hom}(K,\bar{E})}$. An analogous reformulation of (c) will be given in § 4. In the sequel we will often identify a type T like in (d) with the algebraic homomorphism t defined by it. Note that T gives rise to an algebraic homomorphism $K^* \to E^*$ if and only if $n_\sigma = n_{\tau\sigma}$, for every $\tau \in \mathrm{Gal}(\bar{E}/E)$. This is the case if T is the infinity-type of an algebraic Hecke character with values in E.

3. Infinity-types and algebraic Hecke characters

It is not true that, conversely, every algebraic homomorphism $K^* \to E^*$ occurs as infinity-type of an algebraic Hecke character of K with values in E. The first obvious constraint is that such an infinity-type has to kill all totally-positive units $\equiv 1 (\mathrm{mod}\ \mathfrak{f})$ in $\mathcal{O}_K$. As these are of finite index in $\mathcal{O}_K^*$, the proof of Dirichlet's unit theorem implies that there is an integer w such that, for any embedding $\bar{E} \hookrightarrow \mathbb{C}$, inducing an action of complex conjugation, $\sigma \mapsto \bar{\sigma}$, on $\mathrm{Hom}(K,\bar{E})$, and for any $\sigma \in \mathrm{Hom}(K,\bar{E})$, one has

$$(3.1) \qquad n_\sigma + n_{\bar{\sigma}} = w.$$

w is called the weight of T (or of χ).

Thus, for any complex conjugation of $\bar{E}$, we find

$$\chi \cdot \bar{\chi} = \mathbb{N}_{K/\mathbb{Q}}^{w} \quad ,$$

where $\mathbb{N}_{K/\mathbb{Q}}(\mathfrak{a}) = \#(\mathcal{O}_K/\mathfrak{a})$ for an integral ideal $\mathfrak{a}$ of K. (In fact, this is true on a subgroup of finite index of $I_{\mathfrak{f}}$, and $\mathbb{R}_+^*$ is torsion-free.) Therefore the values of an algebraic Hecke character are pure, in the sense that all embeddings into $\mathbb{C}$ have the same absolute value. Similarly, they are what we shall call (for want of a better term) numbers of CM-type: