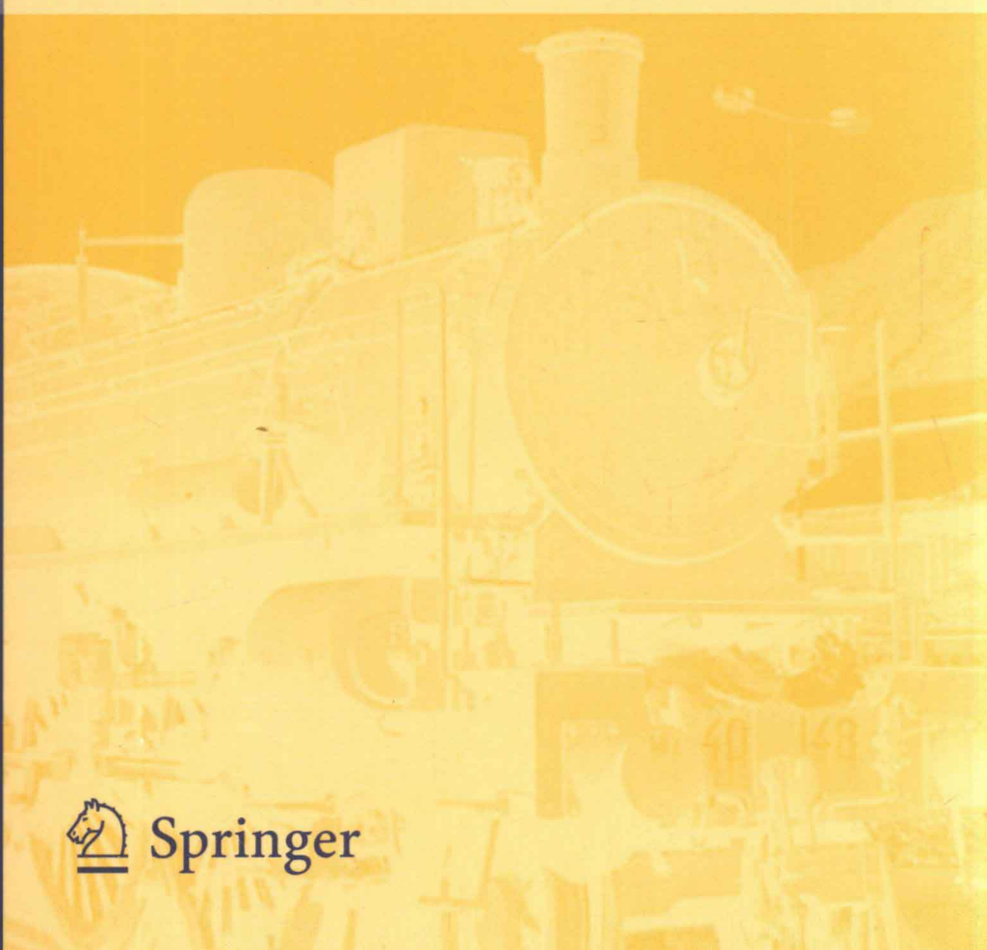
Giuseppe Buttazzo
Aldo Pratelli
Eugene Stepanov
Sergio Solimini

Optimal Urban Networks via Mass Transportation

961



Springer

Giuseppe Buttazzo · Aldo Pratelli
Sergio Solimini · Eugene Stepanov

Optimal Urban Networks via Mass Transportation

Springer

Giuseppe Buttazzo
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5
56127 Pisa
Italy
buttazzo@dm.unipi.it

Aldo Pratelli
Dipartimento di Matematica
Università di Pavia
Via Ferrata 1,
27100 Pavia
Italy
aldo.pratelli@unipv.it

Sergio Solimini
Dipartimento di Matematica
Politecnico di Bari
Via Amendola 126/b
70126 Bari
Italy
solimini@dm.uniba.it

Eugene Stepanov
St. Petersburg University
of Information Technology, Mechanics
and Optics
Kronverkskij pr. 49
197101 St. Petersburg
Russia
stepanov.eugene@gmail.com

ISBN: 978-3-540-85798-3 e-ISBN: 978-3-540-85799-0
DOI: 10.1007/978-3-540-85799-0

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008935629

Mathematics Subject Classification (2000): 49J45, 49Q10, 49Q15, 49Q20, 90B06, 90B10, 90B20

Cover design: SPi Publisher Services

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Lecture Notes in Mathematics 1961

Preface

The monograph is dedicated to a class of models of optimization of transportation networks (urban traffic networks or networks of railroads and highways) in the given geographic area. One assumes that the data on distributions of population and of services/workplaces (i.e. sources and sinks of the network) as well as the costs of movement with and without the help of the network to be constructed, are known. Further, the models take into consideration both the cost of everyday movement of the population and the cost of construction and maintenance of the network, the latter being determined by a given function on the total length of the network. The above data suffice, if one considers optimization in long-term prospective, while for the short-term optimization one also needs to know the transport plan of everyday movements of the population (i.e. the information on "who goes where"). Similar models can also be adapted for the optimization of networks of different nature, like telecommunication, pipeline or drainage networks. In the monograph we study the most general problem settings, namely, when neither the shape nor even the topology of the network to be constructed is a priori known.

To be more precise, given a region $\Omega \subseteq \mathbb{R}^N$, we will model the transportation network to be constructed by an a priori generic Borel set $\Sigma \subseteq \Omega$. We consider then the mass transportation problem in which the paths inside and outside the network Σ are charged differently. The aim is to find the best location for Σ, in order to minimize a suitable cost functional $\mathfrak{F}(\Sigma)$, which is given by the sum of the cost of transportation of the population, and the penalization term depending on the length of the network, which represents the cost of construction and maintenance of the network. To study the problem of existence of optimal solutions, we present first a relaxed version of the optimization problem, where the network is represented by a Borel measure rather than a set, and we prove the existence of a relaxed solution. We will study then the properties of optimal relaxed solutions (measures) and prove that, under suitable assumptions, the relaxed solution solve the original problem, i.e. in fact they correspond to rectifiable sets, and therefore can be called classical solutions. However, it will be shown that in general the problem studied may have no classical solutions. We will also study some topological properties of optimal networks, like closedness and the number of connected components. In particular, we find rather sharp conditions on problem data, which ensure the existence of closed optimal networks and/or optimal networks having at most countably many connected components. Finally, we will prove a general regularity result on optimal networks. Namely, we will show that an optimal network is covered by a finite number of Lipschitz curves of uniformly bounded length, although it may have even uncountably many connected components.

Acknowledgments

This work was conceived during the meeting *Giornate di Lavoro in "Calculus of Variations and Geometric Measure Theory"* held in Levico Terme (Italy), and was carried on thanks to the project "Calcolo delle Variazioni" (PRIN 2004) of the Italian Ministry of Education. The work of the third author was partially supported by the italian GNAMPA–INDAM.

Contents

Chapter 1
Introduction

The present monograph treats one particular class of mathematical models arising in urban planning, namely, the models of optimization of transportation networks such as urban traffic networks, networks of tram or metro lines, railroads or highways. The optimization is performed so as to take into account the known data of the distributions of the population and of services/workplaces (or, more generally, sources and sinks of the network), the costs of the transportation with and without using the network to be constructed, and the budgetary restrictions on construction and maintenance of the network, as well as, in certain cases, the transportation plan of everyday movement of the population. As an illustration, see the distribution of population as well as the railroad network in Italy (Figure 1.1). The functional to be minimized corresponds to the overall cost of everyday transportation of population from their homes to the services together with the cost of construction and maintenance of the network. It is important to emphasize that the shape and even the topology of the network is considered a priori unknown.

From the most general point of view such models belong to the class of economical optimal resource planning problems which were first studied in [44]. In the simplest cases under additional restrictions on the network such problems reduce to problems of minimization of so called average distance functionals (see [20]), and are similar to the well-known discrete problems of optimization of service locations (so-called Fermat-Weber, or k-median problems) studied by many authors (see, e.g. [7, 68, 69, 51]). Similar as well as slightly different models have been proposed for telecommunication, pipeline and drainage networks in [11, 41, 47], and are recently subject to extensive study (see, for instance, [8, 9, 10, 17, 27, 34, 48, 55, 56, 62, 66, 52, 73, 74]. The common kernel of all such models is the general (i.e. not necessarily discrete) setting of the Monge-Kantorovich optimal mass transportation problem (see, e.g. [42, 43, 67, 1, 36, 35, 60, 25, 38]); we give now a short description of the mass transport problem, a more complete discussion is given in Appendix A.

G. Buttazzo et al., *Optimal Urban Networks via Mass Transportation*,
Lecture Notes in Mathematics 1961, DOI: 10.1007/978-3-540-85799-0_1,

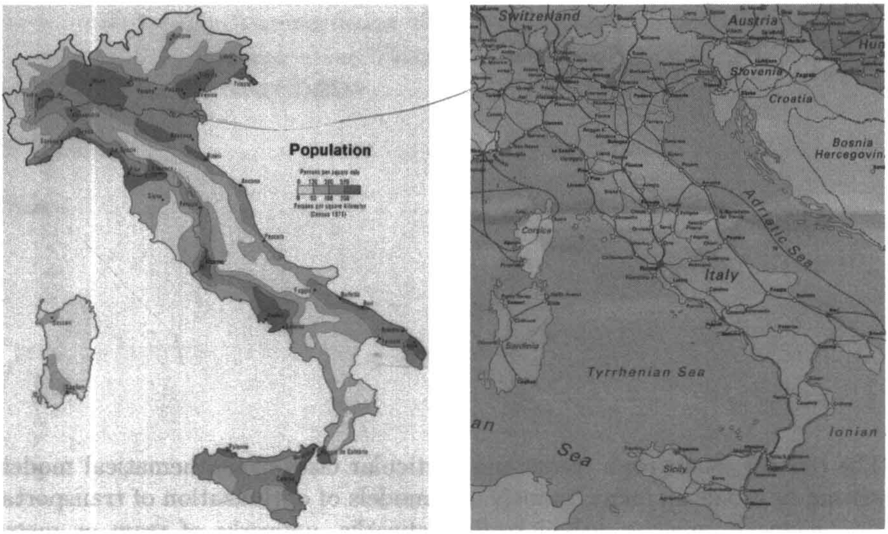


Fig. 1.1 Density of population (left) and railway network (right) in Italy

The mass transportation problem was first proposed by Monge [49]. Using a modern language, this can be restated as follows: we are given a metric space (X, d) and two finite Borel measures f^+ and f^- with the same total mass $\|f^+\| = \|f^-\|$. A Borel map $T : X \to X$ is said to be a *transport map* if it moves f^+ on f^-, that is, if $T_\# f^+ = f^-$ being $T_\#$ the push-forward operator (see Appendix B.2). We are also given the *cost function*, which is a lower semicontinuous function $c : X \times X \to \mathbb{R}^+$; its meaning is very simple, namely $c(x, y)$ is the cost to move a unit mass from x to y. In the original setting of Monge $c(x, y) = d(x, y)$, more generally one is often interested in $c(x, y) = d(x, y)^p$. The Monge transport problem consists then in determining, among all the transport maps, the *optimal transport maps*, that is, those maps which minimize the total transportation cost given by

$$\int_X c\big(x, T(x)\big)\, df^+(x)\,.$$

It may easily happen that there are no transport maps at all, namely when the measure f^+ has singular parts; it may also happen that, even thought there are transport maps, the existence of optimal transport maps fails. Also for this reason, it reveals of primary importance to consider the relaxed form of the problem proposed by Kantorovich (see [42, 43]). The idea of Kantorovich is to define *transport plan* any positive measure γ on $X \times X$ such that the two marginals of γ are precisely f^+ and f^-; the meaning is quite intuitive: such a measure γ is to be interpreted as the strategy of transportation which moves a mass $\gamma\big(\{(x, y)\}\big)$ from x to y; more precisely, it moves a total amount

$\gamma(C \times D)$ of mass from the set C to the set D. An *optimal transport plan*, then, is any transport plan γ minimizing the cost

$$\iint_{X\times X} c(x,y)\, d\gamma(x,y)\,.$$

It is to be noticed that the transport plans are a generalization of the transport maps: indeed, given a transport map T the measure $\gamma_T := (\mathrm{Id}, T)_\# f^+$ is a transport plan, and moreover by definition

$$\iint_{X\times X} c(x,y)\, d\gamma_T(x,y) = \int_X c\big(x, T(x)\big)\, df^+(x)\,;$$

so, the search of optimal plans is a generalization of the search of optimal maps. The power of this new definition is evident: while, as we said, it may happen that there are no transport maps, or no optimal transport maps, there are always transport plans, as for instance $f^+ \otimes f^-$. Moreover, there are always optimal transport plans, since the function c is lower semicontinuous. A more detailed introduction to mass transportation problems is given in Appendix A.

In this monograph we consider a problem of urban planning, in which we take as ambient space a region $\Omega \subseteq \mathbb{R}^N$, with $N \geq 2$ since the one-dimensional case is in fact trivial; the measure f^+ represents the density of the population in the urban area Ω and the measure f^- represents the density of the services or workplaces. We also consider a Borel set $\Sigma \subseteq \Omega$ of finite $\mathcal{H}^1$ length, which represents the urban transportation network that has to be constructed to minimize the cost of transporting f^+ on f^- according to some suitable cost functional.

Once the set Σ is given, the cost $d_\Sigma(x,y)$ to be paid in order to connect any two points x and y of Ω is defined as the least "price" of moving along a Lipschitz curve connecting x and y given by the number

$$\delta_\Sigma(\theta) := A\big(\mathcal{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathcal{H}^1(\theta \cap \Sigma)\big)\,.$$

The functions A and B are two given nondecreasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $A(0) = B(0) = 0$, A being continuous and B lower semicontinuous: $A(s)$ is the "cost" of covering a distance s by own means, that is a number including the expenses for the fuel, the fare of the highway, the fatigue of moving by feet, the time consumption and so on; on the other hand, $B(s)$ represents the cost of covering the distance s making use of the transportation network (i.e. the "cost of the ticket").

In this monograph, we assume the point of view of an "ideal city", where the only goal is to minimize the total expenses for the people; therefore, the number $B(s)$ should be regarded just as a tax that people pay to contribute to the cost of the network when they use it, and the case $B \equiv 0$, corresponding to a situation where everybody can use the public transportation for free, is the simplest (and most common in the literature) choice in this ideal setting.

An opposite point of view, where the owner of the network aims to maximize his total income by choosing a suitable pricing policy B, has been studied in [18].

Having fixed the set Σ, the population will naturally try to minimize its expenses, that is, people choose to move following a transport plan γ minimizing

$$I_\Sigma(\gamma) := \iint_{X \times X} d_\Sigma(x, y)\, d\gamma(x, y)$$

among all admissible transport plans, and we denote by $MK(\Sigma)$ the respective minimum (or the infimum if the minimum is not achieved). We want to find a network Σ minimizing the total cost for the people. However, $MK(\Sigma)$ is not the only cost to be considered: otherwise, a network of infinite length covering the whole Ω would be clearly the optimal choice. We will then consider also a very general cost function $H\big(\mathcal{H}^1(\Sigma)\big)$ for the maintenance of the network, that will depend on the length $\mathcal{H}^1(\Sigma)$ of Σ and that diverges if the length goes to ∞. For instance, one can set

$$H(l) := \begin{cases} 0, & \text{if } l \leq L, \\ +\infty, & \text{if } l > L, \end{cases}$$

which corresponds to a situation where one is allowed to build a network of total length not exceeding L. Our goal is then to find an optimal network Σ_{opt} which minimizes $\Sigma \mapsto MK(\Sigma) + H\big(\mathcal{H}^1(\Sigma)\big)$ among the admissible sets Σ.

The above problem can be considered as a long-term optimization model. In fact, in this case while choosing the optimal network Σ one is allowed to change freely the transportation plan γ (i.e. it is supposed that people may consider it more convenient to choose different destinations for their everyday movements, e.g. change the shops they usually use or even change their workplace, in view of the cost of transportation), which is only reasonable in a quite long-term prospective. On the contrary, the reasonable model for the short-term prospective is obtained by considering given the transport plan γ (i.e. the information on "who goes where" in the everyday movements) and thus minimizing $\Sigma \mapsto I_\Sigma(\gamma) + H\big(\mathcal{H}^1(\Sigma)\big)$ among the admissible sets Σ. However, it is easy to notice, similarly to [18], that the short-term optimization problem is in fact simpler than the long-term one. Hence in this monograph we concentrate on studying the latter with all the results applying also to the former.

Plan of the Monograph

In Chapter 2 we define the general problem setting without additional assumptions on admissible networks. The simplest case, when Σ is a priori required to be connected, will be considered in Chapter 3, and some known facts about this problem will be reported. In this case, by a suitable use of the Hausdorff convergence on connected sets, we show the existence of an optimal network. A particular situation happens when the goal of the planner is simply to transport the source mass f^+ to a network Σ in the most efficient way, that is f^-, instead of being a priori fixed, is chosen in an optimal way among the probabilities with support in Σ. This problem then corresponds to the minimization of the functional

$$F(\Sigma) := \int_\Omega A\big(\mathrm{dist}\,(x, \Sigma)\big)\, df^+(x)\,. \tag{1.1}$$

We will refer to the minimization problem for the functional F defined by (1.1) as the *irrigation problem* in view of the natural interpretation of the cost (1.1) as the total effort to irrigate the mass distribution f^+ using a network Σ. It is assumed that the effort to irrigate the point $x \in \Omega$ depends on its distance t from the network Σ through the function $A(t)$. Taking $A(t) := t$ we have the minimization problem for the *average distance functional*

$$\min\left\{\int_\Omega \mathrm{dist}(x, \Sigma)\, df^+(x) \ :\ \Sigma \subseteq \Omega,\ \Sigma \text{ connected},\ \mathcal{H}^1(\Sigma) \leq L\right\},$$

that has been studied in several recent papers (see, e.g. [17, 21, 19, 20, 54]). On Fig. 1.2 below we show the plot of two cases when Ω is the unit bidimensional disc, f^+ is the Lebesgue measure over Ω, and Σ varies among all connected sets of length L, with two different choices of L.

It is immediate to see that dropping the connectedness assumption leaving the cost functional as in (1.1) would give zero as the minimal value of F, since the set Σ would have the interest to spread everywhere on Ω. This is why the particular situation considered by functional (1.1) is meaningful only in the connected framework.

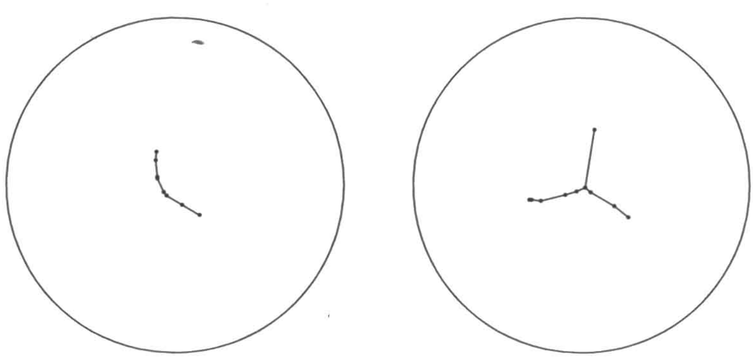


Fig. 1.2 Optimal irrigation networks for $L = 0.5$ (left) and $L = 1$ (right)

In Chapter 4 we show that without extra assumptions on the functions A, B and H there may be no optimal networks. Therefore, we introduce a relaxed version of the problem, where the sets are replaced by Radon measures, and in particular each set Σ corresponds to the measure $\mathscr{H}^1 \llcorner \Sigma$. Then, we show the existence of optimal "relaxed networks", and in particular we prove that optimal measures μ on Ω of the form $\mu = a(x)\mathscr{H}^1 \llcorner \Sigma$ for a one-dimensional rectifiable set Σ and a Borel function $a : \Sigma \to [0, 1]$ always exist. Roughly speaking, this means that there is an optimal transportation network concentrated on a Borel set Σ, but it has a pointwise density in $[0, 1]$: the density 1 corresponds to a standard railway, where covering a path of length l has a cost $B(l)$. In general, covering a path of length l on a network of density $0 \leq p \leq 1$ costs $A\big((1-p)l\big) + B\big(pl\big)$, as if one covers a length pl on the network, and the remaining $(1-p)l$ by own means. Moreover we show that, under suitable assumptions, there are also "classical solutions", that is, optimal networks which naturally correspond to sets (in other words, relaxed solutions with the coefficient $a(x)$ above taking only values 0 and 1). However, we give counterexamples showing that this does not always occur.

In Chapter 5 we consider two questions, namely whether or not there exists an optimal classical network which is closed, or which has only countably many connected components. We present counterexamples to show that this is not always the case, even when classical solutions exist. However, we are able to find conditions under which one has the existence of an optimal classical network that is closed or has countably many connected components.

In Chapter 6 we prove that, under suitable hypotheses, there is a classical optimal network that is covered by a finite number of Lipschitz curves of uniformly bounded length, even if it may still have infinitely many (even more than countably many) connected components.

Finally, the monograph is concluded by two appendices, which present with more details the general mass transportation problem and some tools from Geometric Measure Theory, among which the Disintegration Theorem and the $\Gamma-$convergence, which are used through the volume.

Chapter 2
Problem Setting

In this chapter we introduce the notation and the preliminaries to rigorously set the problem of optimal networks. The formulation in the sense of L. Kantorovich, by using *transport plans*, i.e. measures on the product space $\Omega \times \Omega$, will be presented together with a second equivalent formulation where the main tools are the so-called *transport path measures* that are measures on the family of curves in Ω. This seems to be a very natural formulation that has already been used in previous papers (see for instance [24, 65, 6, 58]) and that allows to obtain in a rather simple way existence results and necessary conditions of optimality.

2.1 Notation and Preliminaries

In this monograph the ambient space Ω is assumed to be a bounded, closed, N–dimensional convex subset of $\mathbb{R}^N$, $N \geq 2$, equipped with the Euclidean distance; the convexity assumption is made here only for simplicity of presentation; in fact, all the results are still valid in the more general case of bounded Lipschitz domains. For any pair of Lipschitz paths $\theta_1, \theta_2 : [0,1] \to \Omega$, we introduce the distance

$$d_\Theta(\theta_1, \theta_2) := \inf \Big\{ \max_{t\in[0,1]} |\theta_1(t) - \theta_2(\varphi(t))| , \quad \varphi : [0,1] \to [0,1] \text{ increasing and bijective} \Big\} , \tag{2.1}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. We define then Θ as the set of the equivalence classes of Lipschitz paths in Ω parametrized over $[0,1]$, where two paths θ_1 and θ_2 are considered equivalent whenever $d_\Theta(\theta_1, \theta_2) = 0$: it is easily noticed that Θ is a separable metric space equipped with the distance d_Θ. Moreover, simple examples show that the infimum in (2.1) might not be attained. It will be often useful to remind that, given any sequence $\{\theta_n\}$

G. Buttazzo et al., *Optimal Urban Networks via Mass Transportation*,
Lecture Notes in Mathematics 1961, DOI: 10.1007/978-3-540-85799-0_2,

of paths in Θ with uniformly bounded Euclidean lengths, by Ascoli–Arzelà Theorem one can find a $\theta \in \Theta$ such that (possibly up to a subsequence) $\theta_n \xrightarrow{d_\Theta} \theta$. This implies, in particular, that the corresponding curves $\theta_n([0,1])$ converge in the Hausdorff distance to $\theta([0,1])$, while the converse implication is not true. Notice that

$$\theta_n \xrightarrow{d_\Theta} \theta \quad \Longrightarrow \quad \mathscr{H}^1\big(\theta([0,1])\big) \leq \liminf_{n\to\infty} \mathscr{H}^1\big(\theta_n([0,1])\big)\,,$$

where $\mathscr{H}^1$ denotes the one-dimensional Hausdorff measure.

In the sequel, for the sake of brevity we will abuse the notation calling θ also the set $\theta([0,1]) \subseteq \Omega$, when not misleading. We call *endpoints* of the path θ the points $\theta(0)$ and $\theta(1)$, and, given two paths $\theta_1, \theta_2 \in \Theta$ such that $\theta_1(1) = \theta_2(0)$, the *composition* $\theta_1 \cdot \theta_2$ is defined by the formula

$$\theta_1 \cdot \theta_2(t) := \begin{cases} \theta_1(2t) & \text{for } 0 \leq t \leq 1/2\,, \\ \theta_2(2t-1) & \text{for } 1/2 \leq t \leq 1\,. \end{cases}$$

As already introduced in Chapter 1, we let now A, $B : \mathbb{R}^+ \to \mathbb{R}^+$ be the costs of moving by own means and by using the network, i.e. $A(s)$ (resp. $B(s)$) is the cost corresponding to a part of the itinerary of length s covered by own means (resp. with the use of the network). This means that, if the urban network is a Borel set $\Sigma \subseteq \Omega$ of finite length, the total cost of covering a path $\theta \in \Theta$ is given by

$$\delta_\Sigma(\theta) := A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big)\,, \tag{2.2}$$

since the length $\mathscr{H}^1(\theta\setminus\Sigma)$ is covered by own means and the length $\mathscr{H}^1(\theta\cap\Sigma)$ is covered by the use of the network. Concerning the functions A and B, we make from now on the following assumptions:

$$A \text{ is nondecreasing, continuous and } A(0) = 0\,; \tag{2.3}$$

$$B \text{ is nondecreasing, l.s.c. and } B(0) = 0\,. \tag{2.4}$$

Note that these hypotheses follow the intuition: the meaning of the assumptions $A(0) = 0$, $B(0) = 0$ and of the monotonicity are obvious, while the continuity of the function A means that a slightly longer path cannot have a much higher cost, and it is a natural assumption once one moves by own means. On the contrary, a continuity assumption on the function B would rule out some of the most common pricing policies which occur in many real life urban transportation networks: for instance, often such a pricing policy is given by a fixed price (the price of a single ticket) for any positive distance, or is a piecewise constant function.

We define now a "distance" on Ω which depends on Σ and is given by the least cost of the paths connecting two points: in short,

$$d_\Sigma(x,y) := \inf\left\{\delta_\Sigma(\theta) :\ \theta \in \Theta,\, \theta(0) = x,\, \theta(1) = y\right\}\,. \tag{2.5}$$