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Edited by A. Dold and B. Eckmann

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PREFACE

These proceedings contain the introductory and specific scientific papers presented at the international

Conference on Multigrid Methods

which was held at Cologne-Porz from 23rd to 27th November, 1981.

The introductory part describes basic methods, theoretical approaches and practical aspects in a systematic way. Furthermore, some simple applications are discussed, and an exemplary multigrid program for a simple model problem is presented. The four authors of the introductory papers have tried to use a uniform notation. This has been rather difficult since in the meantime several systems of notations have come into general use, and good arguments can be found for all of them. The uniformity reached despite of these difficulties does not concern all occurring quantities but the essential ones.

The specific papers deal with the fields of theory, applications and software development. Most studies concern elliptic problems and their solution by means of difference methods. The conference and the papers reflect an increasing interest in the combination of multigrid techniques with defect correction methods as well as in the solution of singularly perturbed and (indefinite) nonlinear problems. Apart from introductory and specific papers this volume also contains a multigrid bibliography.

120 scientists from 12 countries participated in the Conference. Thanks to the generous financial support by the organizing institutions it was not necessary to charge a conference fee.

The organizers are as follows:

- Gesellschaft für Mathematik und Datenverarbeitung (GMD, St. Augustin)
- Sonderforschungsbereich (SFB) 72 "Approximation und mathematische Optimierung" at the University of Bonn, funded by the Deutsche Forschungsgemeinschaft
- Fachausschuß "Effiziente numerische Verfahren für partielle Differentialgleichungen" of the Gesellschaft für Angewandte Mathematik und Mechanik (GAMM).

Within the cooperation with the GMD the Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt (DFVLR) provided the conference rooms. On this occasion the editors like to thank the mentioned institutions and their representatives, Prof. Dr. Norbert Szyperski (chairman of the Executive Board of the GMD), Prof. Dr. Stephan Hildebrandt (speaker of SFB 72) and Prof. Dr. Hermann L. Jordan (chairman of the Executive Board of the DFVLR) for the immaterial and material support of the conference.

The practical organization was carried out by Kurt Brand and Heinz Reutersberg (Institut für Mathematik of the GMD). They were supported by Margarete Donovang, Elisabeth Harf and Reinhild Schwarz. Furthermore, the Abteilung für Informationswesen of the GMD provided substantial assistance to the completion of this volume. We like to express our gratitude to all persons involved.

Finally we like to thank all conference participants and especially the lecturers for their contributions to the success of the conference.

Wolfgang Hackbusch

Ulrich Trottenberg

CONTENTS

MULTIGRID METHODS: FUNDAMENTAL ALGORITHMS, MODEL PROBLEM ANALYSIS AND APPLICATIONS

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Contents

1. Introduction

This paper gives a systematic introduction to multigrid methods for the solution of elliptic differential equations. The paper is based on the two introductory lectures held by the authors on the occasion of the "Conference on Multigrid Methods". It includes basic ideas (Part I) and fundamental methodical approaches (Part II), theoretical approaches (Part III) and simple applications (Part IV). The paper is to be seen in the context of the two other introductory papers in which Wolfgang Hackbusch outlines his general theory of multigrid methods and Achi Brandt gives a guide to the practical realization of multigrid methods. Brandt's paper deals, in particular, with problems of a more general type (systems of differential equations in higher dimensions) than that of the problems we consider in our paper. Brandt also discusses more sophisticated multigrid techniques.

Although our description of the multigrid principle and of the fundamental methodic approaches is quite general, the concrete considerations in this paper refer - in accordance with its introductory character - to a limited class of problems: We explicitly treat only scalar equations in two dimensions; the underlying discretizations are based on finite difference methods. Mostly we are concerned with second order Dirichlet boundary value problems. Most of these restrictions, in particular the restriction to two dimensions, are mainly for the sake of technical simplification.

In this introduction we give a short survey of the development of multigrid methods and on the state of the art (Section 1.1). We will then describe contents and objectives of this paper in some more detail (Section 1.2). In Section 1.3, we will introduce some fundamental notation which is needed.

1.1 Historical remarks and current perspectives

Multigrid history. The multigrid principle (for discrete elliptic boundary value problems) is extremely simple: Approximations with *smooth errors* are obtained very efficiently by applying suitable *relaxation* methods. Because of the error smoothness, corrections of these approximations can be calculated on *coarser grids*. This basic idea can be used *recursively* employing coarser and coarser grids. This leads then to "*(asymptotically) optimal*" iterative methods, i.e. methods for which the computational work required for achieving a fixed accuracy is proportional to the number of discrete unknowns. If the multigrid methods are then combined with the idea of *nested iteration* (use of coarser grids to obtain good initial approximations on finer grids), a suitable algorithmization even yields methods for which the computational work required for achieving the discretisation accuracy is still proportional to the number of discrete unknowns.

Consequently, we may distinguish three elements (stages):

(1) error smoothing by relaxation,

(2) calculation of corrections on coarser grids and recursive application,

(3) combination with nested iteration.

Looking back on the development of multigrid methods we see that the above elements, if considered separately, have already been known or used for a long time before they were combined to efficient multigrid methods. Especially the error smoothing effect of relaxation methods belongs to the classical inventory of numerical knowledge. The idea to use this effect for convergence acceleration can already be found in the early literature (e.g. Southwell [92], [93]; Stiefel [94]). However, the recursive use of coarser grids is not yet elaborate there. But it is only this recursion which gives the above mentioned "optimality".

On the other hand, the recursive application of coarser grids for an efficient solution of specific discrete elliptic boundary value problems was used in the context of "*reduction methods*" introduced by Schröder [86] (see also [85], [87], [88]). Here, however, no explicit error smoothing is performed. Elimination techniques are used instead which transform the original problem "equivalently" to coarser grids. (These elimination techniques restrict the range of direct application of reduction methods to a small class of problems.)

Finally, the self-suggesting idea of nested iterations has in principle been known for a long time.

The first studies introducing and investigating multigrid methods in a narrow sense (elements (1) and (2)) are those by Fedorenko [34], [35] and then that of Bakhvalov [6]. While in [35] Fedorenko restricts the convergence investigation to the Poisson equation in the unit square, Bakhvalov [6] discusses general elliptic boundary value problems of second order with variable coefficients (in the unit square). Bakhvalov also indicates the possibility of combining multigrid methods with nested iteration (element (3)).

Though the studies published by Fedorenko and Bakhvalov have, in principle, shown the asymptotic optimality of the multigrid approach (and to a certain extent its generality as well), their actual efficiency is first recognised only by Achi Brandt (by 1970). Studying adaptive grid refinements and their relation to fast solvers, Brandt has been led to the papers of Fedorenko and Bakhvalov through information given by Olof Widlund. In the first two papers [15], [16] and later on summarised in the systematic work [17], Brandt shows the actual efficiency of multigrid methods. His essential contributions (in the early studies) concern the introduction of non-linear multigrid methods ("FAS-scheme") and adaptive techniques ("MLAT"), the discussion of general domains and local grid refinements, the systematic application of the nested iteration idea ("full multigrid" FMG) and - last but not least - the provision of the tool of the "local Fourier analysis" for theoretical investigation and method optimisation.

Representative for the further multigrid development are the following papers which we would like to mention as being "historically" relevant contributions.

In [4] Astrakhantsev generalises Bakhvalov's convergence result to general boundary conditions; like Bakhvalov he uses a variational formulation in his theoretical approach. - In [39], Frederickson introduces an approximate multigrid-like solver which can be regarded as a forerunner of the "MGR methods", which were developed later on. - After a first study of multigrid methods for Poisson's equation in a square [75], Nicolaides discusses multigrid ideas in connection with finite element discretisations systematically in [76]. -

In the years 1975/76, Hackbusch develops the fundamental elements of multigrid methods anew without having knowledge of the existing literature. It is again Olof Widlund who informs Hackbusch about the studies which are already available. Hackbusch's first systematic report [42] contains many theoretical and practical investigations which have been taken up and developed further by several authors. So one finds considerations of the "model problem analysis" type, the use of "red black" and "four colour" relaxation methods for smoothing, the treatment of non-rectangular domains and of nonlinear problems etc. In the papers [43], [45], [49], Hackbusch then presents a general convergence theory of multigrid methods.

The recent development. Since about 1977 multigrid methods have increasingly gained broad acceptance. This more recent development shall not be described here in detail. (A survey of the literature presently available is given by the multigrid bibliography in this Proceedings.) However, we want to mention some important fields of applications and mathematical areas to which multigrid methods have been applied and extended. The field of finite elements which has first been of a more theoretical interest to multigrid methods (see, for example, [76], [43], [8]) is now undergoing an intensive practical investigation (see, for example, [9], [32]). Apart from linear and non-linear boundary value problems (scalar equations and systems) eigenvalue problems and bifurcation problems (see, for example, [44], [27], [73]) are treated as well. Parabolic (see, for example, [33], [90], [63]) and other time-dependent and non-elliptic problems (see e.g. [23], [22], [84]) are attracting more and more interest. All these situations occur in numerical fluid dynamics, probably the most challenging field for multigrid methods. Here the studies are now concentrating on singular perturbation phenomena, transonic flow, shocks, the treatment of Euler equations and of the full Navier Stokes equations.

Apart from differential equations, integral equations can also be efficiently solved by multigrid methods (see e.g. [25] and the whole complex of multigrid methods "of the second kind" [48], [57]). Furthermore, multigrid-like methods are also being suggested for the solution of special systems of equations without continuous background [25]. A certain amount of multi-level structure (at least the nested iteration idea) can also be found in algorithms used in pattern recognition.

Perhaps as important as the extension of the field of applications of multigrid methods is the combination of the multigrid idea with other numerical and more general mathematical principles. In this context we would like to mention the combination with *extrapolation* and *defect correction methods* (see e.g. [25], [5], [51], [56]). Finally, there are considerations which refer to the optimal use of multigrid methods on *vector* and *parallel computers* (and the construction of corresponding multigrid components) (see, for example, [24]) as well as to approaches within computer architecture concerning a direct mapping of the multigrid principle onto a suitable - perhaps *pyramidal* - multiprocessor structure (see corresponding remarks in [103]).

Delayed acceptance, resentments. The historical survey has shown that the acceptance of multigrid methods was first a rather troublesome process. Only the rapid development of recent years has convinced most people working in the field of numerical methods for partial differential equations of the sensational possibilities provided by the multigrid principle.

Nevertheless, even today's situation is still unsatisfactory in several respects. If this is true for the development of standard methods, it applies all the more to the area of really difficult, complex applications. With respect to standard applications, we would like to discuss this in some detail (since this area is in the center of this introductory paper) and with respect to the complex applications, for example in fluid dynamics, we would like to confine ourselves to some remarks.

As far as standard problems (simple elliptic 2D problems of second order) are concerned, the opinion prevailed for a long time - even and just among experts - that, despite of their "asymptotic optimality", multigrid methods were in reality far from being as efficient as the "direct fast solvers" (such as the Buneman algorithm [29] or the method of total reduction [88]) and their combination with capacitance matrix techniques [81]. Only by providing generally available programs (such as MG00, MG01, see chapter 10), has it been proved in practice that suitable multigrid methods are at least competitive in these areas as well. The decisive advantage of multigrid methods is however that they can be applied easily to problems which do not meet - or do not fully meet - the requirements demanded by direct fast solvers and capacitance matrix techniques.

Doubts in the high efficiency of multigrid methods were also fed by the multigrid convergence theories. The abstract theories are often far too pessimistic and do usually not provide constructive criteria for the construction of optimal methods for concrete situations (see also Section 9.3).Only the *model problem analysis* (see Chapters 3, 7 and 8) and *local Fourier analysis* (see Sections 9.1, 9.2) yield quantitative results to be used for the construction of algorithms. On the other hand, these theoretical approaches, being relatively simple from the mathematical viewpoint, also have disadvantages: The model problem analysis can be applied directly to a small class of problems only, and local Fourier analysis is based on idealising assumptions.

As a consequence, even in the field of standard applications the disagreement about which approach would really supply the "best" or the "most robust" algorithms, is not completely settled as yet. For example, as far as the smoothing methods are concerned, each expert recommends "his" method and emphasises its benefits (A.Brandt recommends standard relaxation techniques - pointwise, linewise and "distributed"; Wesseling the ILU smoothing, Jameson smoothing methods of the ADI type, we recommend MGR methods....). Since so far systematic and fair comparisons were hardly available, it was also impossible, until recently, to obtain reliable statements on which method should be preferred in which situation. Among users this confusion has led to misunderstandings and false conclusions.

While in the field of standard problems the differences in efficiency shown by the various algorithms are, after all, not very large and the disagreement previously mentioned is therefore of a more or less academic nature, the disagreement in the field of non-elementary applications is of direct practical importance and it has especially unpleasant consequences there.

Such a controversy exists, for example, in the field of fluid dynamics between many numerical practitioners who like to take up multigrid methods and multigrid experts (even among the practically oriented experts) who like to develop "optimal" methods from a more fundamental viewpoint. With respect to more complex problems the experts usually supply efficient algorithms for simplified situations only and do not go to the work of solving full fledged industrial problems. The practitioners are therefore sceptical about the full applicability of multigrid methods. They mostly prefer to include single multigrid components in certain parts of available software. Thus, they obtain improvements which are possibly rather impressing, but, on the other hand, they regard their scepsis as being justified since the improvements obtained are far from being as large as predicted for "optimal" methods. However the multigrid experts also feel justified since they regard the stepwise inclusion of multigrid elements in the available "non-multigrid software" as being unsatisfactory in any case. This discrepancy can be found in many publications and comments and it was also reflected on the conference which is the subject of these proceedings. There is some hope, that these proceedings contribute towards bridging the gap between multigrid experts and practitioners.

1.2. Contents of this paper, acknowledgements

In part I, we describe the multigrid idea (Chapter 2) and give a first analysis of a sample method for Poisson's equation. For both chapters we have intentionally chosen a very detailed and elementary representation. The sample method considered in Chapter 3 is a rather inefficient method (since Jacobi relaxation is used for smoothing), but it has the advantage of being particularly theoretically transparent. The theoretical considerations and the tools introduced in Chapter 3 are characteristic for the model problem analysis which is discussed more systematically in part III.

Part II (Chapters 4,5,6) describes the well-known fundamental multigrid techniques: the *recursively defined complete multigrid cycle* (Chapter 4), the *nonlinear full approximation scheme* (Chapter 5), and the *full multigrid method* (Chapter 6).

Parts III and IV, in particular Chapters 7,8 (together with Chapter 3) and 10, 11, inform about results which are largely new and have not been published as yet.

Part III discusses the concepts of the so-called *model problem analysis* and *local Fourier analysis*. For a certain class of model problems and a certain class of multigrid algorithms, it is possible to give exact statements (not estimates) on the convergence behaviour of the method in question using basic tools of discrete Fourier analysis. In Chapter 7, we introduce the required formalism. In this context, various cases of the coarse grid definition are discussed.

Readers who are interested in concrete results rather than in the technically quite complicated formalism should proceed to Chapter 8. All results in this chapter refer to *standard coarsening* (doubling the meshwidths); the emphasis lies on the discussion of efficient smoothing methods, namely on *RB (= red black)*, *ZEBRA*, and *alternating ZEBRA relaxation*. Within the class of methods discussed, the model problem analysis allows the construction of optimal multigrid components.

Problems and methods which can no longer be treated rigorously by model problem analysis may possibly be studied by means of Fourier analysis (Chapter 9). In this context, however, no exact statements on the problem given are obtained but only statements on an idealised problem (and thus on an idealised method) where, in particular, the influence of the boundary and the boundary conditions are neglected. The exact statements on the idealised problem (and method) are then regarded as approximate statements on the original problem (and method). Subjects of this idealizing local Fourier analysis are, for example, the usual *Gauß-Seidel-relaxation* method (with *lexicographic ordering* of the grid points) and *ILU-smoothing*. Among other things, we make a short comparison of ILU-smoothing with ZEBRA relaxation in Section 9.2. - In Section 9.3., we make some remarks on more abstract convergence theories.

On the basis of the model problem and local Fourier analysis, the programs MG00 and MG01 for elliptic "standard problems" have been developed. MG01 is described in Chapter 10. - Chapter 11 describes the possibility of applying multigrid methods in combination with simultaneous use of various coordinate systems to a given problem (composite mesh system).