Statistical Theory of the Analysis of Experimental Design

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Statistical Theory of the Analysis of Experimental Designs

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PREFACE

This book is an outgrowth of the graduate course, "Design of Experiment and Analysis of Variance", given at the University of Calgary, 1970-1973. The book covers the fundamental portions of the statistical theory of the analysis of experimental designs and is strictly restricted to linear models and normal distribution. The author relies heavily on linear algebraic methods as his mathematical tools. The construction of designs is not covered because there are already a couple of excellent books on this subject.

The author was first introduced to this field of modern statistics when he had a fortunate chance to attend Professor R. C. Bose's lectures at the Department of Statistics, University of North Carolina, Chapel Hill, in 1956-1958. Thus the author is heavily indebted to Professor Bose, especially in Chapters IV and V, and Chapter V is a mere reproduction of Professor Bose's leçture. Needless to say, the author is solely responsible for any errors or shortcomings, if any, in the presentation.

The author should acknowledge his indebtness to Professor D. J. Finney's book: *An Introduction to The Theory of Experimental Design*, Chicago University Press, 1963.

One novel point of this book is a fairly satisfactory treatment of the so-called randomization procedure of block designs; the last chapter, VI, is devoted completely to the mathematically rigorous treatment of the randomization of a PBIB design. The author is indebted to his former colleagues Professor S. Ikeda and

Mr. M. Ogasawara for their collaboration in this line of research. Especially the author is grateful to Professor Ikeda, who spent three months during the summer of 1974 with the Department of Mathematics, Statistics and Computing Science, The University of Calgary as a visiting scientist under the sponsorship of National Research Council of Canada through GRANT NO. A 7683, for his painstaking proofreading of the whole manuscript and the addition of the appendix to Chapter 6.

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The author's thanks are due to Miss Barbara Bemben, the editor of the Marcel Dekker, Inc., who took pains to produce this book and also to Mrs. Tae Hayashitani who has done the retyping job of the manuscript.

Calgary, Alberta, Canada.
June, 1974 Junjiro Ogawa

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CONTENTS

Chapter 1

ANALYSIS OF VARIANCE

Since the analysis of variance is the standard technique for handling the data or observations obtained from an experiment, a brief summary of its fundamental features should be presented in the beginning. Readers should make reference to *The Analysis of Variance* by Scheffé [1].

1.1. The Partition of a Sum of Squares Corrected by the Mean

Given n (numerical) objects $a_1, a_2, \ldots, a_n$, the **deviations from the mean**

$$\bar{a} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

are given by

$$a_1 - \bar{a},\ a_2 - \bar{a}, \cdots,\ a_n - \bar{a}.$$

The sum of the squares of the deviations is given by

$$Q = \sum_{\alpha=1}^{n} (a_\alpha - \bar{a})^2 = \sum_{\alpha=1}^{n} a_\alpha^2 - n\bar{a}^2.$$

This is also called *the sum of squares corrected by the mean.*

Let us consider the sum of squares more closely for small values of n:

For $n = 2$,

$$Q = (a_1 - \bar{a})^2 + (a_2 - \bar{a})^2 = a_1^2 + a_2^2 - \frac{1}{2}(a_1 + a_2)^2$$

$$= \frac{1}{2}(a_1 - a_2)^2 = \left(\frac{1}{\sqrt{2}}\, a_1 - \frac{1}{\sqrt{2}}\, a_2\right)^2.$$

For $n = 3$,

$$Q = (a_1 - \bar{a})^2 + (a_2 - \bar{a})^2 + (a_3 - \bar{a})^2$$

$$= a_1^2 + a_2^2 + a_3^2 - \frac{1}{3}(a_1 + a_2 + a_3)^2$$

$$= \frac{1}{2}(a_1 - a_2)^2 + \frac{1}{2}\, a_1^2 + a_1a_2 + \frac{1}{2}\, a_2^2 + a_3^2 - \frac{1}{3}(a_1 + a_2 + a_3)^2$$

$$= \frac{1}{2}(a_1 - a_2)^2 + \frac{1}{6}(a_1^2 + a_2^2 + 4a_3^2 + 2a_1a_2 - 4a_2a_3 - 4a_3a_1)$$

$$= \frac{1}{2}(a_1 - a_2)^2 + \frac{1}{6}(a_1 + a_2 - 2a_3)^2$$

$$= \left(\frac{1}{\sqrt{2}}\, a_1 - \frac{1}{\sqrt{2}}\, a_2\right)^2 + \left(\frac{1}{\sqrt{6}}\, a_1 + \frac{1}{\sqrt{6}}\, a_2 - \frac{2}{\sqrt{6}}\, a_3\right)^2.$$

By a cyclical permutation of the a terms, one obtains the other two partitions:

$$= \left(\frac{1}{\sqrt{2}}\, a_2 - \frac{1}{\sqrt{2}}\, a_3\right)^2 + \left(\frac{-2}{\sqrt{6}}\, a_1 + \frac{1}{\sqrt{6}}\, a_2 + \frac{1}{\sqrt{6}}\, a_3\right)^2$$

$$= \left(\frac{1}{\sqrt{2}}\, a_3 - \frac{1}{\sqrt{2}}\, a_1\right)^2 + \left(\frac{1}{\sqrt{6}}\, a_1 - \frac{2}{\sqrt{6}}\, a_2 + \frac{1}{\sqrt{6}}\, a_3\right)^2.$$

A linear combination of the n objects $a_1, a_2, \ldots, a_n$,

$$L = c_1a_1 + c_2a_2 + \cdots + c_na_n,$$

is said to be a *contrast* of $a_1, a_2, \ldots, a_n$ if $\sum_{\alpha=1}^{n} c_\alpha = 0$. The contrast L is said to be *normalized* if $\sum_{\alpha=1}^{n} c_\alpha^2 = 1$. Two contrasts

$$L_1 = \sum_{\alpha=1}^{n} c_{1\alpha}a_\alpha \quad \text{and} \quad L_2 = \sum_{\alpha=1}^{n} c_{2\alpha}a_\alpha$$

are said to be *mutually orthogonal* if $\sum_{\alpha=1}^{n} c_{1\alpha}c_{2\alpha} = 0$.

For $n = 2$, the sum of squares corrected by the mean is a square of a normalized contrast. For $n = 3$, Q is partitioned into the sum of the squares of two mutually orthogonal, normalized contrasts.

For $n = 4$,

$$Q = (a_1 - \bar{a})^2 + (a_2 - \bar{a})^2 + (a_3 - \bar{a})^2 + (a_4 - \bar{a})^2$$

$$= a_1^2 + a_2^2 + a_3^2 + a_4^2 - \tfrac{1}{4}(a_1 + a_2 + a_3 + a_4)^2$$

$$= \left[a_1^2 + a_2^2 + a_3^2 - \tfrac{1}{3}(a_1 + a_2 + a_3)^2\right]$$

$$+ \left[\tfrac{1}{3}(a_1 + a_2 + a_3)^2 - \tfrac{1}{4}(a_1 + a_2 + a_3 + a_4)^2 + a_4^2\right].$$

Now since

$$\tfrac{1}{3}(a_1 + a_2 + a_3)^2 - \tfrac{1}{4}(a_1 + a_2 + a_3 + a_4)^2 + a_4^2$$

$$= \tfrac{1}{12}(a_1 + a_2 + a_3)^2 - \tfrac{6}{12}(a_1 + a_2 + a_3)a_4 + \tfrac{9}{12}a_4^2$$

$$= \tfrac{1}{12}(a_1 + a_2 + a_3 - 3a_4)^2,$$

one obtains a partition of Q as follows:

$$Q = \left(\frac{1}{\sqrt{2}}a_1 - \frac{1}{\sqrt{2}}a_2\right)^2 + \left(\frac{1}{\sqrt{6}}a_1 + \frac{1}{\sqrt{6}}a_2 - \frac{2}{\sqrt{6}}a_3\right)^2$$

$$+ \left(\frac{1}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 + \frac{1}{\sqrt{12}}a_3 - \frac{3}{\sqrt{12}}a_4\right)^2,$$

and by a cyclical change of the a terms one gets

$$\left(\frac{1}{\sqrt{2}}a_1 - \frac{1}{\sqrt{2}}a_2\right)^2 + \left(\frac{1}{\sqrt{6}}a_1 + \frac{1}{\sqrt{6}}a_2 - \frac{2}{\sqrt{6}}a_4\right)^2$$

$$+ \left(\frac{1}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 - \frac{3}{\sqrt{12}}a_3 + \frac{1}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_1 - \frac{1}{\sqrt{2}}a_3\right)^2 + \left(\frac{1}{\sqrt{6}}a_1 - \frac{2}{\sqrt{6}}a_2 + \frac{1}{\sqrt{6}}a_3\right)^2$$

$$+ \left(\frac{1}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 + \frac{1}{\sqrt{12}}a_3 - \frac{3}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_1 - \frac{1}{\sqrt{2}}a_3\right)^2 + \left(\frac{1}{\sqrt{6}}a_1 + \frac{1}{\sqrt{6}}a_3 - \frac{2}{\sqrt{6}}a_4\right)^2 + \left(\frac{1}{\sqrt{12}}a_1 - \frac{3}{\sqrt{12}}a_2 + \frac{1}{\sqrt{12}}a_3 + \frac{1}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_1 - \frac{1}{\sqrt{2}}a_4\right)^2 + \left(\frac{1}{\sqrt{6}}a_1 - \frac{2}{\sqrt{6}}a_2 + \frac{1}{\sqrt{6}}a_4\right)^2 + \left(\frac{1}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 - \frac{3}{\sqrt{12}}a_3 + \frac{1}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_1 - \frac{1}{\sqrt{2}}a_4\right)^2 + \left(\frac{1}{\sqrt{6}}a_1 - \frac{2}{\sqrt{6}}a_3 + \frac{1}{\sqrt{6}}a_4\right)^2 + \left(\frac{1}{\sqrt{12}}a_1 - \frac{3}{\sqrt{12}}a_2 + \frac{1}{\sqrt{12}}a_3 + \frac{1}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_2 - \frac{1}{\sqrt{2}}a_3\right)^2 + \left(\frac{-2}{\sqrt{6}}a_1 + \frac{1}{\sqrt{6}}a_2 + \frac{1}{\sqrt{6}}a_3\right)^2 + \left(\frac{1}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 + \frac{1}{\sqrt{12}}a_3 - \frac{3}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_2 - \frac{1}{\sqrt{2}}a_3\right)^2 + \left(\frac{1}{\sqrt{6}}a_2 + \frac{1}{\sqrt{6}}a_3 - \frac{2}{\sqrt{6}}a_4\right)^2 + \left(\frac{-3}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 + \frac{1}{\sqrt{12}}a_3 + \frac{1}{\sqrt{12}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}a_2 - \frac{1}{\sqrt{2}}a_4\right)^2 + \left(\frac{-2}{\sqrt{6}}a_1 + \frac{1}{\sqrt{6}}a_2 + \frac{1}{\sqrt{6}}a_4\right)^2 + \left(\frac{1}{\sqrt{12}}a_1 + \frac{1}{\sqrt{12}}a_2 - \frac{3}{\sqrt{12}}a_3 + \frac{1}{\sqrt{12}}a_4\right)^2,$$

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$$\left(\frac{1}{\sqrt{2}}\,a_2 - \frac{1}{\sqrt{2}}\,a_4\right)^2 + \left(\frac{1}{\sqrt{6}}\,a_2 - \frac{2}{\sqrt{6}}\,a_3 + \frac{1}{\sqrt{6}}\,a_4\right)^2$$

$$+ \left(\frac{-3}{\sqrt{12}}\,a_1 + \frac{1}{\sqrt{12}}\,a_2 + \frac{1}{\sqrt{12}}\,a_3 + \frac{1}{\sqrt{12}}\,a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}\,a_3 - \frac{1}{\sqrt{2}}\,a_4\right)^2 + \left(\frac{-2}{\sqrt{6}}\,a_1 + \frac{1}{\sqrt{6}}\,a_3 + \frac{1}{\sqrt{6}}\,a_4\right)^2$$

$$\left(\frac{1}{\sqrt{12}}\,a_1 - \frac{3}{\sqrt{12}}\,a_2 + \frac{1}{\sqrt{12}}\,a_3 + \frac{1}{\sqrt{12}}\,a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}\,a_3 - \frac{1}{\sqrt{2}}\,a_4\right)^2 + \left(\frac{-2}{\sqrt{6}}\,a_2 + \frac{1}{\sqrt{6}}\,a_3 + \frac{1}{\sqrt{6}}\,a_4\right)^2$$

$$+ \left(\frac{-3}{\sqrt{12}}\,a_1 + \frac{1}{\sqrt{12}}\,a_2 + \frac{1}{\sqrt{12}}\,a_3 + \frac{1}{\sqrt{12}}\,a_4\right)^2.$$

These are all sums of the squares of three orthonormal contrasts. Indeed the three contrasts in the last partitions are

$$0a_1 + 0a_2 + \frac{1}{\sqrt{2}}\,a_3 - \frac{1}{\sqrt{2}}\,a_4, \quad 0a_1 - \frac{2}{\sqrt{6}}\,a_2 + \frac{1}{\sqrt{6}}\,a_3 + \frac{1}{\sqrt{6}}\,a_4,$$

$$-\frac{3}{\sqrt{12}}\,a_1 + \frac{1}{\sqrt{12}}\,a_2 + \frac{1}{\sqrt{12}}\,a_3 + \frac{1}{\sqrt{12}}\,a_4.$$

One can see that there are other types of partition, as follows:

$$\left(\frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_2\right)^2 + \left(\frac{1}{\sqrt{2}}\,a_3 - \frac{1}{\sqrt{2}}\,a_4\right)^2$$

$$+ \left(\frac{1}{\sqrt{4}}\,a_1 + \frac{1}{\sqrt{4}}\,a_2 - \frac{1}{\sqrt{4}}\,a_3 - \frac{1}{\sqrt{4}}\,a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_3\right)^2 + \left(\frac{1}{\sqrt{2}}\,a_2 - \frac{1}{\sqrt{2}}\,a_4\right)^2$$

$$+ \left(\frac{1}{\sqrt{4}}\,a_1 - \frac{1}{\sqrt{4}}\,a_2 + \frac{1}{\sqrt{4}}\,a_3 - \frac{1}{\sqrt{4}}\,a_4\right)^2,$$

$$\left(-\frac{3}{\sqrt{20}}a_1 - \frac{1}{\sqrt{20}}a_2 + \frac{1}{\sqrt{20}}a_3 + \frac{3}{\sqrt{20}}a_4\right)^2 + \left(\frac{1}{\sqrt{4}}a_1 - \frac{1}{\sqrt{4}}a_2 - \frac{1}{\sqrt{4}}a_3 + \frac{1}{\sqrt{4}}a_4\right)^2$$
$$+ \left(-\frac{1}{\sqrt{20}}a_1 + \frac{3}{\sqrt{20}}a_2 - \frac{3}{\sqrt{20}}a_3 + \frac{1}{\sqrt{20}}a_4\right)^2,$$

$$\left(\frac{1}{\sqrt{4}}a_1 + \frac{1}{\sqrt{4}}a_2 - \frac{1}{\sqrt{4}}a_3 - \frac{1}{\sqrt{4}}a_4\right)^2 + \left(\frac{1}{\sqrt{4}}a_1 - \frac{1}{\sqrt{4}}a_2 + \frac{1}{\sqrt{4}}a_3 - \frac{1}{\sqrt{4}}a_4\right)^2$$
$$+ \left(\frac{1}{\sqrt{4}}a_1 - \frac{1}{\sqrt{4}}a_2 - \frac{1}{\sqrt{4}}a_3 + \frac{1}{\sqrt{4}}a_4\right)^2.$$

In this case we notice that

$$\begin{Vmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{\sqrt{4.3}} & \frac{1}{\sqrt{4.3}} & \frac{1}{\sqrt{4.3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3.2}} & \frac{1}{\sqrt{3.2}} \\ 0 & 0 & -\sqrt{\frac{1}{2}} & \frac{1}{\sqrt{2.1}} \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \end{Vmatrix}$$

are orthogonal matrices. The former can be generalized for any n, but the latter can be generalized only for $n \equiv 0 \bmod 4$.

So far we have seen that *the sum of the squares of* n *deviations from the mean is partitioned into the sum of* $(n - 1)$ *squares of mutually orthonormal contrasts* for $n = 2, 3,$ and 4. One can show that this is generally true. In fact, if we put

$$\sqrt{n}\,\bar{a} = \frac{1}{\sqrt{n}}(a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n),$$

$$L_1 = \sqrt{\frac{n-1}{n}}\,a_1 - \frac{1}{\sqrt{n(n-1)}}(a_2 + \cdots + a_n),$$

$$L_2 = \sqrt{\frac{n-2}{n-1}}\,a_2 - \frac{1}{\sqrt{(n-1)(n-2)}}(a_3 + \cdots + a_n),$$

. . . .

$$L_\alpha = \sqrt{\frac{n-\alpha}{n-\alpha+1}}\,a_\alpha - \frac{1}{\sqrt{(n-\alpha+1)(n-\alpha)}}(a_{\alpha+1} + \cdots + a_n),$$

. . . .

$$L_{n-1} = \frac{1}{\sqrt{2}}(a_{n-1} - a_n),$$

it is easy to check that this is an orthogonal transformation, called *the Helmert transformation*, and hence

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n\bar{a}^2 + L_1^2 + \cdots + L_{n-1}^2 .$$

and consequently

$$Q = a_1^2 + \cdots + a_n^2 - n\bar{a}^2 = L_1^2 + L_2^2 + \cdots + L_{n-1}^2 .$$

If we take any orthogonal matrix of order $n - 1$

$$H = \| h_{\alpha\beta} \|,$$

and let

$$L'_\alpha = \sum_{\beta=1}^{n-1} h_{\alpha\beta} L_\beta \qquad (\alpha = 1, \ldots, n-1),$$

then

$$\sum_{\alpha=1}^{n-1} L_\alpha^2 = \sum_{\alpha=1}^{n-1} L'^2_\alpha .$$

Thus, although there are so many choices of the mutually orthogonal normalized contrasts, the number of the orthogonal contrasts is the dimension of the linear space

$$\sum c_\alpha = 0,$$

which is called *the contrast space.* It is invariant and is known as *the degree of freedom* of the sum of squares Q.

Each contrast is said *to carry a degree of freedom.* Degrees of freedom carried by orthogonal contrasts are said *to be orthogonal.*

Suppose the given n objects are divided into p groups with elements $r_1,\ldots,r_p$, respectively,

$$r_1 + r_2 + \cdots + r_p = n.$$

Let the subtotals of the groups be $U_1, U_2, \ldots, U_p$, and hence

$$U_1 + U_2 + \cdots + U_p = n\bar{a}.$$

Then *the sum of squares*

$$\sum_{j=1}^{p} \frac{U_j^2}{r_j} - \frac{(\Sigma_j U_j)^2}{n} = \sum_{j=1}^{p} \frac{U_j^2}{r_j} - n\bar{a}^2$$

carries $(p - 1)$ *degrees of freedom, and the difference*

$$\sum_{\alpha=1}^{n} a_\alpha^2 - \sum_{j=1}^{p} \frac{U_j^2}{r_j}$$

carries $(n - p)$ *degrees of freedom, which are orthogonal to the degrees of freedom carried by the sum of squares* $\Sigma_j U_j^2/r_j - n\bar{a}^2$.

Proof. Let $\underline{\zeta}_j' = (\zeta_{j1},\ldots,\zeta_{jn})$, where $\zeta_{jf} = 1$ if a_f is in the jth group and 0 otherwise; then

$$\underline{\zeta}_j'\underline{\zeta}_j = r_j \text{ and } \underline{\zeta}_i'\underline{\zeta}_j = 0 \qquad (i \neq j),$$

$$\sum_{j=1}^{p} \underline{\zeta}_j = \underline{1}$$

$$U_j = \underline{\zeta}_j'\underline{a} \qquad (j = 1,\ldots,p).$$

One can construct $r_j - 1$ mutually orthogonal and normalized contrasts that are linear combinations only of those a terms for which $\zeta_{jf} = 1$.

There are $\sum_{j=1}^{p} (r_j - 1) = n - p$ such contrasts that are orthogonal to those contrasts that are linear combinations of $U_1,\ldots,U_p$.

Now let

$$W_j = \sqrt{r_j}\left(\frac{U_j}{r_j} - \frac{U_1 + \cdots + U_p}{n}\right),$$

then

$$\sum_{j=1}^{p} \sqrt{r_j}\, W_j = 0$$

and

$$\sum_{j=1}^{p} W_j^2 = \sum_{j=1}^{p} r_j \left(\frac{U_j}{r_j} - \frac{U_1 + \cdots + U_p}{n}\right)^2 = \sum_{j=1}^{p} \frac{U_j^2}{r_j} - \frac{(U_1 + \cdots + U_p)^2}{n}.$$

Therefore the sum of the squares of $U_1,\ldots,U_p$ can be written as

$$\sum_{j=1}^{p} \frac{U_j^2}{r_j} - \frac{(U_1 + \cdots + U_p)^2}{n}$$

$$= W_1^2 + \cdots + W_{p-1}^2 + \frac{1}{r_p}\left(\sum_{j=1}^{p-1} \sqrt{r_j}\, W_j\right)^2$$

$$= (W_1,\ldots,W_{p-1}) \left\Vert \begin{matrix} 1 + \frac{r_1}{r_p} & \frac{\sqrt{r_1 r_2}}{r_p} & \cdots & \frac{\sqrt{r_1 r_{p-1}}}{r_p} \\ \frac{\sqrt{r_1 r_2}}{r_p} & 1 + \frac{r_2}{r_p} & \cdots & \frac{r_2 r_{p-1}}{r_p} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\sqrt{r_1 r_{p-1}}}{r_p} & \frac{\sqrt{r_2 r_{p-1}}}{r_p} & \cdots & 1 + \frac{r_{p-1}}{r_p} \end{matrix} \right\Vert \begin{pmatrix} W_1 \\ \vdots \\ W_{p-1} \end{pmatrix}.$$

If one makes the transformation

$$L_1 = \sqrt{\frac{r_2 + \cdots + r_p}{r_1 + r_2 + \cdots + r_p}}\, W_1$$

$$- \frac{\sqrt{r_1}}{\sqrt{(r_1 + \cdots + r_p)(r_2 + \cdots + r_p)}} \left(\sqrt{r_2}\, W_2 + \cdots + \sqrt{r_p}\, W_p\right)$$

$$= \sqrt{\frac{r_1 + r_2 + \cdots + r_p}{r_2 + \cdots + r_p}}\, W_1,$$

$$L_2 = \sqrt{\frac{r_3 + \cdots + r_p}{r_2 + r_3 + \cdots + r_p}}\, W_2$$

$$- \frac{\sqrt{r_2}}{\sqrt{(r_2 + \cdots + r_p)(r_3 + \cdots + r_p)}} \left(\sqrt{r_3}\, W_3 + \cdots + \sqrt{r_p}\, W_p\right)$$

$$= \frac{\sqrt{r_1 r_2}}{\sqrt{(r_2 + \cdots + r_p)(r_3 + \cdots + r_p)}}\, W_1 + \sqrt{\frac{r_2 + r_3 + \cdots + r_p}{r_3 + \cdots + r_p}}\, W_2,$$

$$L_3 = \frac{\sqrt{r_1 r_3}}{\sqrt{(r_3 + \cdots + r_p)(r_4 + \cdots + r_p)}}\, W_1$$

$$+ \frac{\sqrt{r_2 r_3}}{\sqrt{(r_3 + \cdots + r_p)(r_4 + \cdots + r_p)}}\, W_2$$

$$+ \sqrt{\frac{r_3 + r_4 + \cdots + r_p}{r_4 + \cdots + r_p}}\, W_3,$$

. . . .

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