Contemporary Concepts in Physics
Volume 3

A.M. Polyakov

Gauge Fields and Strings

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Gauge Fields and Strings

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Preface to the Series

The series of volumes, *Concepts in Contemporary Physics*, is addressed to the professional physicist and to the serious graduate student of physics. The subjects to be covered will include those at the forefront of current research. It is anticipated that the various volumes in the series will be rigorous and complete in their treatment, supplying the intellectual tools necessary for the appreciation of the present status of the areas under consideration and providing the framework upon which future developments may be based.

Preface

For many years I have been keeping notes on different topics in physics—a kind of scientific diary. They contain occasional new results and mostly derivations of known things, done in a way that seemed nice to me. The notes were very helpful when I needed to recall some subject. It is surely best to consult with one's own self.

This book has arisen from these notes, or better to say, from the part of them devoted to field theory. I decided to publish it because it seems that there are some people who may find it useful.

In many cases I discuss things that have never been completely understood. I do this in the hope that the approach I suggest, although imperfect, will stimulate deeper penetration into the subject.

I do not give many references in this book (except for very recent results). The reason is that although to study the history of physics and to distribute credits is an interesting enterprise, I am not yet prepared for it.

The reader can find extra information and references in many review papers, e.g. J. Kogut and K.G. Wilson, *Physics Reports*, **12**, 75–199 (1974); J. Kogut, *Reviews of Modern Physics*, **55**, 775–836 (1983); A.A. Migdal, *Physics Reports*, **102**, 199–290 (1983); Patashinsky, Pokrovsky, "Fluctuation Theory of Phase Transitions" Pergamon Press, Oxford (1979) and "Superstrings" (J. Schwarz ed.), World Scientific Pub (1985).

Also, below I list (in arbitrary order) some of my favorite papers that had a profound influence on this book. The choice is, by definition, subjective and incomplete:

A. M. Polyakov

1. A. Patashinsky and V. Pokrovsky, *Zhetph* **46**, 994 (1964).
2. V. Gribov and A. Migdal, *Zhetph* **55**, 1498 (1968).
3. V. Vaks and A. Larkin, *Zhetph* **49**, 975 (1975).
4. V. Berezinsky, *Zhetph* **61**, 1144 (1971).
5. K. Wilson, *Phys. Rev. D* **10**, 2445 (1974).
6. M. Gell-Mann and F. Low, *Phys. Rev.* **111**, 582 (1954).

7. L. Landau, A. Abrikosov, and I. Khalatnikov, *Dan.* **95**, 497 (1954).
8. L. Faddeev and V. Popov, *Phys. Lett. B* **25**, 29 (1967).
9. T. Skyrme, *Proc. Roy. Soc. London Section A* **260**, 127 (1961).
10. J. Schwinger, *Phys. Rev.* **94**, 1362 (1954).
11. M. Atiyah, V. Pathody, and I. Singer, *Math. Proc. Camb. Phys. Soc.* **77**, 43 (1975).
12. R. Jackiw and K. Rebbi, *Phys. Rev. D* **14**, 517 (1971).
13. J. Kogut and L. Susskind, *Phys. Rev. D* **11**, 395 (1975).
14. G. t'Hooft, *Phys. Rev. Lett.* **37**, 8 (1976).
15. G. t'Hooft, *Nucl. Phys. B* **72**, 461 (1974).
16. L. Brink, P. DiVecchia, and P. Howe, *Phys. Lett. B* **65**, 471 (1976).
17. S. Deser and B. Zumino, *Phys. Lett. B* **65**, 369 (1976).
18. C. Marshall and P. Raymond, *Nucl. Phys. B* **85**, 375 (1975).
19. M. Green and J. Schwartz, *Nucl. Phys. B* **255**, 93 (1985).
20. A. Migdal, *Nucl. Phys. B* **180**, 71 (1981).
21. D. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1345 (1973).
22. H. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).
23. A. Zamolodchikov, *Zhetph Lett.* **17**, 28 (1986).
24. K. Wilson, *Phys. Rev.* **25**, 2003 (1969).
25. D. Amati and M. Testa, *Phys. Lett. B* **48**, 227 (1974).

CONTENTS

CHAPTER 1

Statistical Mechanics and Quantum Field Theory

1.1 Quantum Particles

We have no better way of describing elementary particles than quantum field theory. A quantum field in general is an assembly of an infinite number of interacting harmonic oscillators. Excitations of such oscillators are associated with particles. The special importance of the harmonic oscillator follows from the fact that its excitation spectrum is additive, i.e. if E_1 and E_2 are energy levels above the ground state then $E_1 + E_2$ will be an energy level as well. It is precisely this property that we expect to be true for a system of elementary particles. Therefore we attempt to identify the Hamiltonian of the particles with the Hamiltonian of coupled oscillators (there is a familiar example from solid state physics: the excitations of a crystal lattice can be interpreted as particles—phonons). All this has the flavour of the XIX century, when people tried to construct mechanical models for all phenomena. I see nothing wrong with it because any nontrivial idea is in a certain sense correct. The garbage of the past often becomes the treasure of the present (and *vice versa*). For this reason we shall boldly investigate all possible analogies together with our main problem.

A very important analogy, which will be extensively used below, is the one between the quantum mechanics of a $\mathscr{D}$-dimensional system and the classical statistical mechanics of a $\mathscr{D} + 1$-dimensional system. Let us demonstrate it in the simplest case of the $\mathscr{D} = 1$ quantum mechanics of one particle. According to the Feynman principle, the transition amplitude F from the point x to the point x' is given by the sum over all possible trajectories connecting points x and x', each trajectory entering with the weight $\exp((i/\hbar)S[x(t)])$ where $S[x(t)]$ is the classical action. Therefore:

$$F(x, x', T) = \int\limits_{\substack{x(0)=x \\ x(T)=x'}} \mathscr{D}x(t) \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m\dot{x}^2}{2} - v(x(t))\right] dt\right\} \tag{1.1}$$

Here F is the amplitude, T is the time allowed for the transition, $v(x)$ is an external potential, and the functional integral is defined in the following way. Split the interval $[0, T]$ into N small pieces $[0, t_1]$, $[t_1, t_2], \ldots, [t_{N-1}, T]$. Consider instead of (1.1) the expression:

$$F = \int \prod_{j=1}^{N-1} dx_j \left(\frac{m}{2\pi i\hbar(t_j - t_{j-1})}\right)^{1/2} \left(\frac{m}{2\pi i\hbar(T - t_{N-1})}\right)^{1/2}$$

$$\times \exp \frac{i}{\hbar}\left\{\sum_{j=1}^{N} \frac{m(x_j - x_{j-1})^2}{2(t_j - t_{j-1})} - \sum_{j=1}^{N} (t_j - t_{j-1}) v(x_{j-1})\right\} \tag{1.2}$$

(here $x_0 = x$, $t_0 = 0$, $x_N = x'$, $t_N = T$).

Now, it is possible to show that as the mesh $t_{j+1} - t_j \sim T/N \to 0$ the expression (1.2) has a finite limit that is precisely the transition amplitude. While I do not intend to prove it (and refer instead to the book by Feynman and Hibbs), I shall explain briefly the origin of the formulae (1.1) and (1.2). It is actually quite simple. According to standard quantum mechanics, the transition amplitude is given by†:

$$F(x, x', T) = \langle x' | e^{-(i/\hbar)HT} | x \rangle \tag{1.3}$$

where H is the Hamiltonian. We can rewrite (1.3) in the following way:

$$F(x, x', T) = \langle x' | e^{-(i/\hbar)H(T - t_{N-1})} e^{-(i/\hbar)H(t_{N-1} - t_{N-2})} \cdots e^{-(i/\hbar)Ht_1} | x \rangle$$

$$= \int \langle x' | e^{-(i/\hbar)H(T - t_{N-1})} | x_{N-1} \rangle \langle x_{N-1} | e^{-iH(t_{N-1} - t_{N-2})/\hbar} | x_{N-2} \rangle$$

$$\times \cdots \times \langle x_1 | e^{-(i/\hbar)Ht_1} | x \rangle \, dx_{N-1} \cdots dx_1 \tag{1.4}$$

It is easy to check that as all time intervals $t_{j+1} - t_j \to 0$ we obtain:

$$\langle x_{j+1} | e^{-(i/\hbar)H(t_{j+1} - t_j)} | x_j \rangle \underset{t_{j+1} - t_j \to 0}{\rightarrow}$$

$$\left(2\pi i \frac{\hbar}{m}(t_{j+1} - t_j)\right)^{-1/2} \exp \frac{i}{\hbar}\left\{\frac{m}{2}\frac{(x_{j+1} - x_j)^2}{t_{j+1} - t_j} - v(x_j)(t_{j+1} - t_j)\right\} \tag{1.5}$$

† We put $\langle x' | x \rangle = \delta(x' - x)$

After substitution of (1.5) into (1.4) we obtain (1.2). Notice also that without the potential v the formula (1.5) is exact for any value $t_{j+1} - t_j$ and describes the propagation of a free particle.

In order to establish the analogy with classical statistical mechanics one has to consider the propagation for imaginary time T. Namely, let us look at

$$Z(x, x', T) = \langle x'|e^{-(HT/\hbar)}|x\rangle = F(x, x', -iT) \tag{1.6}$$

We can repeat the splitting procedure again with the only difference that the t_j in (1.4) will acquire an extra factor $-$i. In this way we obtain:

$$Z(x, x', T) = \int\limits_{\substack{x(0)=x \\ x(T)=x'}} \mathscr{D}x(t) \exp\left\{-\frac{1}{\hbar}\int_0^T \left(\frac{m}{2}\dot{x}^2 + v(x(t))\right) dt\right\} \tag{1.7}$$

which is to be understood in the same way as (1.1). The mnemonic rule for passing from (1.1) to (1.7) is very simple: consider the expression:

$$i\int_0^{iT} \left(\frac{m}{2}\dot{x}^2 - v(x(t))\right) dt \tag{1.8}$$

and introduce $t = -i\tau$. We obtain:

$$(1.8) = -\int_0^T \left\{\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + v(x(\tau))\right\} d\tau \tag{1.9}$$

The derivation (1.9) shows also that we have even more freedom in computing the functional integral. Namely, we can chose the splitting points $\{t_j\}$ to lie on an arbitrary contour in the complex plane, and therefore time not only can be imaginary but also can go along some complex path (provided that the convergence condition for (1.5), Im $\Delta t < 0$, is satisfied). For some problems this freedom is very useful. At the moment, however, we are interested in a different aspect of all this. Namely, that formula (1.9) has an important physical interpretation. Let us consider an elastic string of length T and tension m with the ends fixed at x and x'. Suppose that this string stays in an external potential $v(x)$. The potential energy of such a string will be given by:

$$\mathscr{E}_{\text{pot}}[x(t)] = \int_0^T \left\{\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + v(x(\tau))\right\} d\tau \tag{1.10}$$

Notice, again, that now τ is *not* a time but the length of the elastic string. According to the Boltzmann principle, the classical partition function of the string is proportional to:

$$Z \sim \int \mathscr{D}x(t) e^{-\beta \mathscr{E}_{\text{pot}}[x(t)]} \tag{1.11}$$

(β being the inverse temperature), (we have omitted the contribution from the kinetic energy, since in classical statistical mechanics it factors out and does not depend on x and x'). Comparison of (1.11) and (1.9) shows the first analogy between classical statistical mechanics and quantum mechanics: *The transition amplitude for a quantum particle for the time* $(-iT)$ *is equal to the classical partition function for a string of length* T *computed at the value of* $\beta = 1/\hbar$.

The second analogy follows from the fact that the *quantum* partition function for the particle is given by $Z_{\text{qu}} = \operatorname{Tr} e^{-\beta H}$ and hence:

$$Z_{\text{qu}} = \int \mathrm{d}x\, F(x, x, -i\beta\hbar) \tag{1.12}$$

Therefore our second rule is that *in the quantum case the inverse temperature acts as imaginary time.*

Our derivation of these analogies was technical. I feel that there are deep reasons for them, connected with the properties of space-time. Although no real explanation exists, I shall give some comments on this below, when discussing gravity. At the moment our aims are more modest—we are going to exploit these analogies in concrete problems. It is quite clear that, although we have derived everything for one particle, both of our analogies are true for an arbitrary number of degrees of freedom.

1.2 . Global and Local Symmetries. Preliminary Description

Elementary particles existing in nature resemble very much excitations of some complicated medium (ether). We do not know the detailed structure of the ether but we have learned a lot about effective lagrangians for its low energy excitations. It is as if we knew nothing about the molecular structure of some liquid but did know the Navier–Stokes equation and could thus predict many exciting things. Clearly, there are lots of different possibilities at the molecular level leading to the same low energy picture. For theoretical purposes we can take any model we like if it has desirable low energy properties.

In this section we shall discuss the most fundamental symmetry properties of particle physics in context of such specially chosen models. Perhaps the most important discovery of modern particle physics is the gauge principle. According to it, all interactions in nature arise from the claim that the Lagrangian has to be invariant under local symmetry transformations, i.e. symmetry rotations that may be different at different space-time points. It is remarkable that this claim predicts the low energy structure of the Lagrangian.

The first (and most complicated) example of this phenomenon was general relativity, in which, due to the presence of the gravitational field, it is possible to perform Lorentz rotations, different at each point. The second (and easiest) example was quantum electrodynamics, in which the gauge group is abelian (the arbitrariness of the phase of the electron wave function). And lastly, we have the Yang–Mills fields, which are supposed to mediate strong and weak interactions. The study of the dynamics of gauge fields is the most important problem of modern physics.

Using the analogies described in the preceeding section, we shall first examine certain classical systems, and then formulate results in the language of particle theory.

1.3 Discrete Global Symmetries

Let us begin with the case of global (nongauge) symmetries. The simplest example is the well-known Ising model. Its partition function is given by:

$$\begin{aligned} Z &= \sum_{\{\sigma_x\}} \mathrm{e}^{-\beta\mathscr{E}[\sigma_x]} \\ \mathscr{E}[\sigma_x] &= -\sum_{(x,\boldsymbol{\delta})} \sigma_x \sigma_{x+\boldsymbol{\delta}} \end{aligned} \tag{1.13}$$

Here x denotes a site of a cubic lattice, $\boldsymbol{\delta}$ is a unit vector connecting this site with one of its nearest neighbours and the variable σ_x is ± 1. It is clear that the system is invariant under the Z_2 group: $\sigma_x \to -\sigma_x$. If the dimensionality of the x space is more than 1, the system (1.13) has two different phases. In the high temperature (small β) phase the Z_2 symmetry is unbroken and we do not have long range order. By that I mean that if one considers a large but finite system and fixes the value of σ_x at the boundary B by the condition:

$$\sigma_x|_{x \in B} = 1 \tag{1.14}$$

we have the average value of $\langle \sigma_x \rangle$ inside the system vanishing as the size of the system goes to infinity. To prove this, let us compute the correlation function in the small β limit. We have:

$$\begin{aligned}\langle \sigma_0 \sigma_R \rangle &= Z^{-1}\left(\sum_{\sigma_x} e^{\beta \Sigma \sigma_x \sigma_{x+\delta}} \sigma_0 \sigma_R \right) \\ &\approx \beta^{|R|} Z^{-1} \left(\sum_{\sigma_x} \sigma_0 (\sigma_0 \sigma_\delta) \cdots (\sigma_{R-\delta} \sigma_R) \sigma_R \right) \\ &= \beta^{|R|} \end{aligned} \tag{1.15}$$

(In (1.15) we have expanded the exponent in β and left the lowest nonvanishing order obtained by the string of $\beta(\sigma_x, \sigma_{x+\delta})$ along the shortest path connecting the points 0 and $\boldsymbol{R}$). We conclude that, since the correlation length is small, being of the order $(\log(1/\beta))^{-1}$, the influence of the boundary condition inside the system must also be small. So, one expects that for small β:

$$\langle \sigma_x \rangle \sim e^{-L \log(1/\beta)} \underset{L \to \infty}{\to} 0 \tag{1.16}$$

(L being the size of the system).

Now let us look at the case of large β (low temperature phase). The maximal contribution to (1.13) in this case will be given by the configuration with all $\sigma_x = 1$. The probability for a spin to flip is of the order of $e^{-4\mathscr{D}\beta}$, so one expects:

$$\langle \sigma_x \rangle = 1 - O(e^{-4\mathscr{D}\beta}) \tag{1.17}$$

Here $\mathscr{D}$ is the dimensionality of space and $2\mathscr{D}$ is equal to the number of nearest neighbours. However, (1.17) is not completely true. For $\mathscr{D} = 1$ the entropy effects spoil the order completely for all β. In order to see how this happens, let us examine a one dimensional Ising chain. In the ground state all the spins point up. The general configuration can be described by marking the links that connect opposite spins. If there are n such links, then the energy factor of the system is just $e^{-2\beta n}$ but the number of such configurations is $2(N!/n!(N-n)!)$. (N is the total number of links). As a result:

$$Z = \sum_n 2 \frac{N!}{n!(N-n)!} e^{-2\beta n} \tag{1.18}$$