Higher Mathematics

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a collection of problems

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На английском языке

ВЫСШАЯ МАТЕМАТИКА

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ЗАДАЧНИК

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Preface

This collection of problems has been compiled to suit the textbooks entering into our series "Higher Mathematics". These textbooks are referred to in the present collection as:

[1] – *Differential and Integral Calculus;*
[2] – *Fundamentals of Linear Algebra and Analytical Geometry;*
[3] – *Differential Equations. Multiple Integrals. Series. Theory of Functions of a Complex Variable*

Indicated at the beginning of each section of the present book are the chapter and section from the above mentioned textbooks where the relevant theoretical material can be found.

As a rule, each section contains a minimum number of problems which corresponds to the number of teaching periods assigned to study a certain topic. The odd-numbered problems may be recommended to be solved in auditorium, while the even-numbered problems must be solved by students at home independently.

The problems found in textbooks [1]-[3] may also be used for practical studies. These are not contained in the present Collection.

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Chapter 1

INTRODUCTION TO ANALYSIS

Sec. 1.1. Real Numbers. Sets

Using the method of mathematical induction, prove the following relationships:

1. $1+2+3+ \ldots +n = n(n+1)/2$.

2. $1^2+2^2+ \ldots +n^2 = n(n+1)(2n+1)/6$.

3. $(1+x)^n \geqslant 1+nx, \quad x > -1$.

4. $\frac{1}{2} \cdot \frac{3}{4} \ldots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

To solve the below problems, it is necessary to study Chapter 1 from [1].

5. Let the set A consist of the youths of a given group, and the set B of the girls of the same group. Find $A \cup B$, $A \cap B$, $A \backslash B$. Also, consider the case when A or B is an empty set.

6. Let $A = \{2n\}$, $B = \{2n+1\}$. Find $A+B$, AB, $A \backslash B$ (n natural).

7. Which number is greater, a or b: $a = 1.(1234512)$, $b = 1.(12345)$; $a = 1.(12302)$, $b = 1.(123)$; $a = 1.(123412)$, $b = 1.(1234)$?

8. Find out to what number a the sequence of real numbers

$$a_1 = 0.1010101010\ldots,$$
$$a_2 = 0.1100110011\ldots,$$
$$a_3 = 0.111000111000\ldots,$$
$$a_4 = 0.111100001111\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n = 0.\underbrace{11\ldots1}_{n \text{ times}}\underbrace{00\ldots0}_{n \text{ times}}1\ldots10\ldots0\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

is stabilized.

9. Find the sum of real numbers $a = 0.(12)$ and $b = 0.(13)$.

10. Given the sets $A = [2, 5]$ and $B = (3, 6)$. Find $A+B$, AB, $A \backslash B$.

11. Solve the following inequalities:

(a) $|x+3| < 0.1$; (b) $|x-3| \geqslant 10$;

(c) $|x| > |x+3|$; (d) $|3x-1| < |x-1|$;

(e) $\left|\frac{x-2}{3x+1}\right| \leqslant 1$.

12. Which of the two numbers is greater: a or $(-a)$?

13. Let $a \geqslant 0$. For what numbers b do the following relations take place

(a) $|a+b| = |a|+|b|$;

(b) $|a-b| = |a|+|b|$;

(c) $|a+b| < |a|+|b|$;

(d) $|a-b| < |a|+|b|$?

14. Find the modulus of the given number: (a) $\ln(1/e)$; (b) $\sin(3\pi/2)$, (c) $\cos(7\pi/4)$.

Sec. 1.2. Limit of a Sequence (See [1], Chapter 2)

15. Prove that

$$\lim_{n\to\infty}\frac{n+1}{n} = 1,$$

and determine for every $\varepsilon > 0$ the number $n_0 = n_0(\varepsilon)$ such that

$$\left|\frac{n+1}{n}-1\right| < \varepsilon \quad \text{if} \quad n > n_0.$$

Fill in the table:

ε	0.1	0.001	0.00001	...
n_0				

In problems 16 to 19 find the indicated limits.

16. $\lim\limits_{n\to\infty}\frac{10^6 n}{n^2+1}$. **17.** $\lim\limits_{n\to\infty}\frac{n^\alpha \sin(n!)}{n+1}$ $(0 \leqslant \alpha < 1)$.

18. $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)$.

19. $\lim\limits_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3}\right)$.

20. Prove that the variable α_n is an infinitesimal if

$$\alpha_n = \frac{n}{n^3+1}; \qquad \alpha_n = \frac{1}{n!}; \qquad \alpha_n = \frac{1}{\sqrt{n^2+1}}.$$

21. Prove that the variable β_n is an infinitely large quantity if

$$\beta_n = (-1)^n n^2; \qquad \beta_n = 2^{\sqrt{n}}; \qquad \beta_n = \ln(n+1).$$

22. Will the sequence $x_n = n^{(-1)^n/2}$ be infinitely large?
In Problems 23 and 24 prove the given equalities.

23. $\lim\limits_{n\to\infty} (\sqrt{3n+10} - \sqrt{3n}) = 0.$

24. $\lim\limits_{n\to\infty} \dfrac{3n^2+5}{2n^2+3n+1} = \dfrac{3}{2}.$

Using the theorem on existence of the limit of a monotone sequence, prove that the following sequences are convergent (Problems 25 and 26):

25. $x_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\dots\left(1-\frac{1}{2^n}\right).$

26. $x_n = \frac{2}{1}\cdot\frac{3}{3}\cdot\frac{4}{5}\cdot\frac{5}{7}\dots\frac{n+1}{2n-1}.$

27. Find the largest term of the following sequences:

$$x_n = \frac{n^2}{2^n}; \qquad x_n = \frac{\sqrt{n+1}}{10+n}.$$

28. Find the smallest term of the following sequences:

$$x_n = \left(1+\frac{1}{n}\right)^n; \qquad x_n = n^2 - 9n - 10.$$

29. Find $\inf x_n$, $\sup x_n (n \in \mathbf{N})$, $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ if

$$x_n = 1 - \frac{1}{n}; \qquad x_n = \frac{(-1)^n}{n+1} + \frac{2+(-1)^n}{3} \quad (n \in \mathbf{N}).$$

30. Which numbers are the partial limits of the sequence

$$1, \frac{1}{2}, -1, 1, \frac{1}{3}, -1, 1, \frac{1}{4}, -1, 1, \dots ?$$

A *partial limit* of an arbitrary bounded sequence is understood as the limit of its convergent subsequence. The existence of such subsequences in a bounded sequence is implied by the Bolzano-Weierstrass theorem.

Applying Cauchy's test for convergence, prove that the following sequences are convergent (Problems 31 to 33):

31. $x_n = \frac{\sin 1^2}{2} + \frac{\sin 2^2}{2^2} + \ldots + \frac{\sin n^2}{2^n} \equiv \sum_{k=1}^{n} \frac{\sin k^2}{2^k}$.

32. $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. **33.** $x_n = \sum_{k=1}^{n} \frac{\cos k!}{k(k+1)}$.

Sec. 1.3. Functions. The Limit of a Function (See [1], Chapter 3)

Find the domain E of definition of the function $y = f(x)$ and the image $E_1 = f(E)$ of the set E with the aid of the function f (Problems 34 and 35).

34. $y = \frac{1}{1+x^2}$. **35.** $y = \sqrt{2+3x-x^2}$.

36. Find $f(0)$, $f(x+2)$, $f(1/x)$, $f(x)+1$, $1/f(x)$ if

$$f(x) = \frac{1-x^2}{1+x^2}.$$

In Problems 37 to 41 find the graph of the given function.

37. $y = 8x-2x^2$. **38.** $y = \frac{1-x^2}{1+x^2}$.

39. $y = -x^2+2x-1$. **40.** $y = \frac{1-x}{1+x}$. **41.** $y = \frac{3x+4}{4x-3}$.

42. Determine the lower and the upper bound to the set of values of the function $f(x)$ if

$$f(x) = x^2 \quad \text{on} \quad [-2, 5]; \quad \varphi(x) = x+\frac{1}{x} \quad \text{on} \quad (0, 3].$$

Hint. On the set $(0, 3]$ $\varphi(x) \geqslant 2$.

43. Construct the graphs of the functions

$$f(x) = \sup_{0 \leqslant t \leqslant x} \{\sin t\}; \quad \varphi(x) = \inf_{0 \leqslant t \leqslant x} \{\sin t\}.$$

In Problems 44 to 48 find the limits of the indicated functions.

44. (a) $\lim_{x \to 0} \frac{x^2-1}{2x^2-x-1}$; (b) $\lim_{x \to 1} \frac{x^2-1}{2x^2-x-1}$;

(c) $\lim_{x \to \infty} \frac{x^2-1}{2x^2-x-1}$; (d) $\lim_{x \to 0} \frac{(1+x)^3-(1+3x)}{x^2+3x^3}$;

(e) $\lim_{x \to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$; (f) $\lim_{x \to \infty} \left(\frac{x^2+3x-1}{2x^2-x+4}\right)^x$;

(g) $\lim\limits_{x \to 3} \dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$; (h) $\lim\limits_{x \to -2} \dfrac{\sqrt[3]{x-6}+2}{x^3+8}$;

(i) $\lim\limits_{x \to a} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$;

(j) $\lim\limits_{x \to +\infty} [\sin\sqrt{x+1}-\sin\sqrt{x}]$.

45. (a) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$; (b) $\lim\limits_{x \to 0} \dfrac{1-\cos 2x}{x^2}$; (c) $\lim\limits_{x \to 0} \dfrac{\tan 4x}{x}$.

46. $\lim\limits_{x \to 0+0} \dfrac{\sin\sqrt{x}}{x}$.

47. (a) $\lim\limits_{x \to \infty} \left(1+\dfrac{2}{x}\right)^{2x}$; (b) $\lim\limits_{x \to 0} (1+3x)^{1/x}$;

(c) $\lim\limits_{x \to \pi/2} (\sin x)^{\tan x}$;

48. (a) $\lim\limits_{x \to 0} \dfrac{\ln(1+4x)}{x}$; (b) $\lim\limits_{x \to b} \dfrac{a^x-a^b}{x-b}$ $(a > 0)$.

In Problems 49 to 56 investigate the given functions for continuity, represent the functions graphically, and determine the character of the points of discontinuity.

49. $f(x) = |x-1|$. **50.** $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1, \\ A, & x = 1. \end{cases}$

51. $f(x) = \operatorname{sgn}(x^2-2x-3)$. **52.** $y = \dfrac{1+x}{1+x^3}$.

53. $y = \dfrac{x}{1+x}$. **54.** $y = \operatorname{sgn}(\cos x)$.

55. $y = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 2-x, & 1 < x \leq 2. \end{cases}$ **56.** $f(x) = \begin{cases} \cos x, & x \leq 0, \\ a+x, & x > 0. \end{cases}$

In Problems 57 and 58 prove that the functions $f(x)$ are uniformly continuous on $[a, b]$, i.e. $\forall \varepsilon > 0$ there is $\delta(\varepsilon) > 0$ (independent of the points of the interval) such that

$$|f(x_1)-f(x_2)| < \varepsilon,$$

whenever

$$|x_1-x_2| < \delta.$$

57. $f(x) = \begin{cases} x^2, & 0 \leq x \leq 2, \\ 20-8x, & 2 \leq x \leq 3. \end{cases}$

58. $f(x) = x^3$, $0 \leq x \leq 2$.

59. Find the inverse of the function

$$y = \frac{ax+b}{cx+d} \quad (ad-bc \neq 0).$$

Let $x \to 0$. Separate the leading term of the form Ax^m (Problems 60 and 61).

60. $f(x) = 3x+x^4$. **61.** $f(x) = \sqrt{1+x}-\sqrt{1-x}$.

Sec. 1.4. Derivatives (See [1], Chapter 4)

62. Find $f'(0)$, $f'(2)$, if $f(x) = 2-2x+x^3$.

63. Find $f'(0)$, $f'(1)$, if $f(x) = x \arcsin \frac{x}{x+1}$.

In Problems 64 to 71 find the derivatives of the given functions.

64. $y = \frac{2x}{1-x^2}$.

65. (a) $y = x+\sqrt[3]{x}$; (b) $y = \frac{1}{x}+\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}$;

(c) $y = \frac{x}{(1-x)^2(1+x)^3}$; (d) $y = x\sqrt{1+x^2}$;

(e) $y = \frac{x}{\sqrt{a^2-x^2}}$; (f) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$;

(g) $y = \sqrt[3]{x^2}-\frac{2}{\sqrt{x}}$; (h) $y = \tan\frac{x}{2}-\cot\frac{x}{2}$;

(i) $y = \frac{\sin x - x\cos x}{\cos x + x\sin x}$; (j) $y = \frac{1}{\cos^n x}$; (h) $y = 2^{\tan\frac{1}{x}}$;

(l) $y = e^{e^x}+e^{e^{ex}}$; (m) $y = x^{a^a}+a^{x^a}$ $(a > 0)$;

(n) $y = \arcsin\frac{1-x^2}{1+x^2}$; (o) $y = \arccos\sqrt{1-x^2}$.

66. $y = \tan x+\frac{1}{3}\tan^3 x$. **67.** $y = e^{-x^2}$.

68. (a) $y = \sin x^2$; (b) $y = \sin^2 x$; (c) $y = \sin^3 x^7$;

(d) $y = \cos(\sin x)$; (e) $y = \cos x^2$; (f) $y = \cos^2 x^4$.

69. (a) $y = \arcsin(x/a)$; (b) $y = \arctan(x/a)$;

(c) $y = \ln(x+\sqrt{a^2+x^2})$; (d) $y = \arcsin(\sin x)$;

(e) $y = \arccos(x/a)$, (f) $y = e^{x^3+x}$.

70. $y = \ln\tan(x/2)$.

71. (a) $y = x \arctan x$; (b) $y = \ln^3 x^2$;

(c) $y = \ln(\ln(\ln x))$; (d) $y = \frac{1}{4}\ln\frac{x^2-1}{x^2+1}$;

(e) $y = \sqrt{x+1} - \ln(1+\sqrt{x+1})$; (f) $y = \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$;

(g) $y = \frac{1}{x}(\ln^3 x + 3\ln^2 x + 6\ln x + 6)$;

(h) $y = \frac{3}{2}(1-\sqrt[3]{1+x^2})^2 + 3\ln(1+\sqrt[3]{1+x^2})$;

(i) $y = \sqrt{x} - \arctan\sqrt{x}$; (j) $y = \arctan\frac{1+x}{1-x}$;

(k) $y = \frac{1}{12}\ln\frac{x^4-x^2+1}{(x^2+1)^2} - \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}}{2x^2-1}$;

(l) $y = \frac{x^6}{1+x^{12}} + \arctan x^6$; (m) $y = \arctan(\tan^2 x)$;

(n) $y = x\arctan x - 0.5\ln(1+x^2) - 0.5(\arctan x)^2$;

(o) $y = \arctan(x+\sqrt{1+x^2})$.

(p) There takes place the formula

$$\frac{d}{dx}\begin{vmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \dots & \dots & \dots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{vmatrix} = \sum_{k=1}^{n}\begin{vmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \dots & \dots & \dots \\ a_{k-1,1}(x) & \dots & a_{k-1,n}(x) \\ a'_{k1}(x) & \dots & a'_{kn}(x) \\ a_{k+1,1}(x) & \dots & a_{k+1,n}(x) \\ \dots & \dots & \dots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{vmatrix},$$

where the elements of the determinant $a_{ij}(x)$ are differentiable functions. Hence, the derivative of a determinant of order n is equal to the sum of n determinants of order n each of which differs from the original determinant by that the appropriate row in it is replaced by the row made up from the derivatives of the elements of the replaced row.

Prove the differentiation formula for determinants of the second and third order.

In Problems 72 and 73 find the derivatives and construct the graphs of the given functions and their derivatives.

72. $y = \begin{cases} 1-x, & -2 < x < 1, \\ (1-x)(2-x), & 1 \leqslant x \leqslant 2, \\ -(2-x), & 2 < x < 4. \end{cases}$

73. $y = \begin{cases} x, & x < 0, \\ \ln(1+x), & x \geqslant 0. \end{cases}$

Find the logarithmic derivatives (i.e. y'/y) of the indicated functions y (Problems 74 and 75).

74. $y = x\sqrt{\dfrac{1-x}{1+x}}$.

75. $y = \cosh^2 x$.

In Problems 76 to 79 find the derivatives of the indicated hyperbolic functions.

76. (a) $y = \sinh(x^2+1)$; (b) $y = \sinh^3 x^6$.

77. $y = \cosh^2(x^2+x+1)$.

78. (a) $y = \tanh^2 x$; (b) $y = \tanh x^2$.

79. (a) $y = \tanh(\ln x+1)$; (b) $y = \operatorname{Arsinh} x$;

(c) $y = \operatorname{Arsinh}(x+\sqrt{1+x^2})$; (d) $y = \ln\sinh x$;

(e) $y = \cosh\ln x$; (f) $y = e^{\tanh x}$;

(g) $y = (\sinh x)^{\cosh x}$ $(x > 0)$; (h) $y = \dfrac{1}{\cosh^3 x}$;

(i) $y = \tanh\dfrac{x}{2} - \coth\dfrac{x}{2}$.

80. For the function $f(x) = x^2+x+1$ determine the differential and the increment at the point $x = 1$ for $\Delta x = 0.1$.

In Problems 81 to 84 find the differentials of the given functions.

81. $d(xe^x)$.

82. $d(\sinh x)$.

83. $d(\sinh x - x\cosh x)$.

84. $d(\ln(1-x^2))$.

85. Find the second-order derivative of the following functions: (a) $y = e^{-x^2} \equiv \exp(-x^2)$; (b) $y = x\sqrt{1+x^2}$.

86. Let there be given the Wronskian

$$W(x) = \begin{vmatrix} y_1(x) & \dots & y_n(x) \\ y_1'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots \\ y_1^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix},$$

where the functions $y_1(x), \dots, y_n(x)$ are continuous on (a, b) together with their derivatives up to order n inclusive.