Number 390

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Memoirs
of the American Mathematical Society

Providence • Rhode Island • USA

September 1988 • Volume 75 • Number 390 (second of 3 numbers) • ISSN 0065-9266

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Published by the
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island, USA

September 1988 • Volume 75 • Number 390 (second of 3 numbers)

1980 *Mathematics Subject Classification* (1985 *Revision*).
Primary 20E28, 20G05, 20G15.

Library of Congress Cataloging-in-Publication Data

Testerman, Donna M., 1960-
Irreducible subgroups of exceptional algebraic groups/Donna M. Testerman.
p. cm. - (Memoirs of the American Mathematical Society, ISSN 0065-9266; no. 390)
"September 1988. Volume 75. Number 390 (end of volume)."
Bibliography: p.
ISBN 0-8218-2453-8
1. Linear algebraic groups. 2. Representations of groups. 3. Embeddings (Mathematics)
I. Title. II. Series
QA3.A57 no. 390
[QA171]
510 s-dc19 88-15554
[512'.22] CIP

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SUBSCRIPTION INFORMATION. The 1988 subscription begins with Number 379 and consists of six mailings, each containing one or more numbers. Subscription prices for 1988 are $239 list, $191 institutional member. A late charge of 10% of the subscription price will be imposed on orders received from nonmembers after January 1 of the subscription year. Subscribers outside the United States and India must pay a postage surcharge of $25; subscribers in India must pay a postage surcharge of $43. Each number may be ordered separately; *please specify number* when ordering an individual number. For prices and titles of recently released numbers, see the New Publications sections of the NOTICES of the American Mathematical Society.

BACK NUMBER INFORMATION. For back issues see the AMS Catalogue of Publications.

MEMOIRS of the American Mathematical Society (ISSN 0065-9266) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, Rhode Island 02904. Second Class postage paid at Providence, Rhode Island 02940. Postmaster: Send address changes to Memoirs of the American Mathematical Society, American Mathematical Society, Box 6248, Providence, RI 02940.

ABSTRACT

Let Y be a simply connected, simple algebraic group of exceptional type, defined over an algebraically closed field k of characteristic $p > 0$. The main result describes all semisimple, closed connected subgroups of Y which act irreducibly on some rational kY module V. This extends work of Dynkin who obtained a similar classification for algebraically closed fields of characteristic 0. The main result has been combined with work of G. Seitz to obtain a classification of the maximal closed connected subgroups of the classical algebraic groups defined over k.

AMS subject classification (1980).

20E28, 20G05, 20G15.

Key words and phrases.

exceptional algebraic groups, representation theory, embeddings of algebraic groups

CONTENTS

INTRODUCTION

Our purpose here is to study triples (A,Y,V), where Y is a simply connected, simple algebraic group of exceptional type, defined over an algebraically closed field k of characteristic $p > 0$, V is an irreducible rational kY module, and A is a semisimple, closed connected subgroup of Y such that $V|A$ is irreducible. (We refer to the above set of hypotheses as the "main problem.") In our main result, we obtain a precise description of the triples (A,Y,V).

Before stating our result, we introduce the following notation. Let T_A be a maximal torus of A, T_Y a maximal torus of Y, with $T_A \leq T_Y$. Let $\Pi(A) = \{\alpha_1, \alpha_2, \ldots\}$ and $\Pi(Y) = \{\beta_1, \beta_2, \ldots\}$ be bases of the root systems $\Sigma(A)$ and $\Sigma(Y)$, respectively, with μ_i the fundamental dominant weight corresponding to α_i and λ_i the fundamental dominant weight corresponding to β_i. Let λ be the high weight of V. (Our labelling of Dynkin diagrams is described on page 8.) Finally, we write $A = G_2$, for example, to mean that $\Sigma(A)$ has type G_2.

Main Theorem. If $V|Y$ is tensor indecomposable, one of the following holds:

(i) $A = A_1$, $Y = G_2$, $\lambda|T_A = 6\mu_1$, $\lambda|T_Y = \lambda_1$ and $p \geq 7$.

(ii) $Y = G_2$, $p=3$, $\Sigma(A)$ is a subsystem of $\Sigma(Y)$ containing all long (respectively, short) roots of $\Sigma(Y)$, and $\lambda|T_Y$ has long (short) support.

Received by the editors February 8, 1987 and, in revised form September 4, 1987.

(iii) $A = G_2$, $Y = F_4$, $p=7$, and $\lambda|T_A = 2\mu_1$ and $\lambda|T_Y = \lambda_4$.

(iv) $Y = F_4$, $p=2$, $\Sigma(A)$ is a subsystem of $\Sigma(Y)$ containing all long (respectively, short) roots, and $\lambda|T_Y$ has long (short) support.

(v) $A = A_2$, $Y = E_6$, $\lambda|T_A = 2\mu_1+2\mu_2$, $\lambda|T_Y = \lambda_1$ or λ_6, and $p\neq 2,5$.

(vi) $A = G_2$, $Y = E_6$, $\lambda|T_A = 2\mu_1$, $\lambda|T_Y = \lambda_1$ or λ_6, and $p\neq 2,7$.

(vii) $A = C_4$, $Y = E_6$, $\lambda|T_A = \mu_2$, $\lambda|T_Y = \lambda_1$ or λ_6, and $p\neq 2$.

(viii) $Y = E_6$, $A = F_4$ is the fixed point subgroup of the graph automorphism of Y and

(a) $\lambda|T_Y = \lambda_1 + (p-2)\lambda_3$ or $(p-2)\lambda_5 + \lambda_6$, for $p>2$, or

(b) $\lambda|T_Y = (p-3)\lambda_1$ or $(p-3)\lambda_6$, for $p>3$.

Moreover, if the pair (A,Y) is as in (ii), (iv) or (viii) $V|A$ is irreducible. As well, if $p\geq 7$ (respectively, $p = 7$, $p \neq 2,5$, $p \neq 2,7$, $p \neq 2$) and Y has type G_2 (respectively, F_4, E_6, E_6, E_6), there exists a subgroup $B \leq Y$, of type A_1 (respectively, G_2, A_2, G_2, C_4) such that B acts irreducibly on $V(\lambda_1)$ (respectively, $V(\lambda_4)$, $V(\lambda_1)$, $V(\lambda_1)$, $V(\lambda_1)$) with the high weight described in (i) (respectively, (iii), (v), (vi), (vii)).□

The results of (i), (ii) and (iv) are proven in [12], where G. Seitz considered the main problem in case Y is a classical group. We establish (iii), (v), (vi) (vii) and (viii) and the existence of an irreducible A_1 in G_2 in this paper. The proof of the existence of an irreducible C_4 in E_6 was communicated to the author by Seitz and is also included here. The remaining existence proofs ($A_2 < E_6$, $G_2 < E_6$ and $G_2 < F_4$) are given in [16], where the conjugacy classes of the irreducible subgroups are also determined.

For an arbitrary irreducible rational kY module V, Steinberg's tensor product theorem ([15]) implies $V|Y = V_1^{q_1}\otimes\cdots\otimes V_k^{q_k}$, where each V_i is a nontrivial irreducible kY module with restricted high weight and $\{q_1,\ldots,q_k\}$ are distinct p-powers. (We refer to $V_i^{q_i}$ as a conjugate of V_i.) If $V|A$ is irreducible, for some subgroup A, then $V_i|A$ is irreducible for each i and the triple (A,Y,V_i) is described in the above theorem. Hence, there is no loss of generality in assuming throughout that $V|Y$ is tensor indecomposable.

The consideration of triples (A,Y,V) in the case where char(k) = 0 was undertaken by E.B. Dynkin in [7]. Given A, a semisimple algebraic group and $\varphi: A \longrightarrow SL(V)$ an irreducible rational representation, Dynkin determined all overgroups of A in SL(V), Sp(V) or SO(V). In a straightforward way, this information yielded a classification of all maximal, proper, closed connected subgroups of the classical algebraic groups. In our situation, where char(k) = p, the Main Theorem has been combined with the results obtained by Seitz in [12] to obtain a similar classification of the maximal proper closed connected subgroups of the classical algebraic groups over k. (This is perhaps the most striking application of the results to date.)

Theorem (A). Let A be a simple algebraic group and $\varphi: A \longrightarrow SL(V)$ an irreducible, rational representaion which is tensor indecomposable. Then with specified exceptions, the image of A is maximal among proper, closed connected subgroups in one of SL(V), Sp(V) or SO(V). Moreover, any other maximal, proper closed connected subgroup of the isometry group of V arises naturally as the stabilizer of a subspace of V or the stabilizer of a tensor product decomposition of V.□

For a more precise statement and the proof, see Theorem (3) in [12]. By far, the major portion of the proof of Theorem (A) lies in describing the "specified exceptions." These fall into two categories, as follows:

Theorem (B). Let Y be a simple algebraic group and $\varphi: Y \rightarrow SL(V)$ an irreducible rational representation which is tensor indecomposable. If A is a proper, closed connected subgroup of Y with $V|\varphi(A)$ irreducible, then one of the following holds:

(i) $\varphi(Y) = SL(V)$, $Sp(V)$ or $SO(V)$.

(ii) $(\varphi(A),\varphi(Y),V)$ appears in Table 1.□

Table 1 contains the combined results of this paper and [12], and lists all embeddings $A < Y < SL(V)$ where A and Y are irreducible, $V|Y$ is tensor indecomposable and $Y \neq Sp(V)$ or $SO(V)$. For a complete explanation of the notation in Table 1, see the end of the introduction; we make a few remarks here. To describe the modules $V|A$ and $V|Y$, we give the high weights. To describe the embedding of A in Y, we indicate the action of a covering group of A on the irreducible kY module W, where W is the natural, classical module for Y, if Y is classical, and W is an irreducible, restricted kY module of minimal dimension, if Y is exceptional. Finally, we note that there are examples for arbitrarily large primes for which there are no counterparts in the characteristic zero result; e.g. I_1' and T_1 in Table 3. Hence, interestingly enough, the philosophy that the answer to the sort of question studied here should be the same for large primes p as the answer to the analogous zero characteristic question fails to be justified.

The methods in [12] and this paper differ greatly from those of Dynkin, by necessity. Since in characteristic p, rational modules for simple groups need not be completely reducible nor tensor indecomposable, as in zero characteristic, some of Dynkin's key reductions do not carry over. Though we may assume that in the triple (A,Y,V), $V|Y$ is tensor indecomposable, it happens that $V|A$ can be tensor decomposable. One may notice that throughout the paper, case-by-case analysis is required whenever this possibility persists. If we desired only to prove Theorem (A) or to give a new proof of Dynkin's result, we could assume $V|A$ to be tensor indecomposable and shorten much of our work. As well, for a new proof of the zero characteristic result the small prime analysis of Chapter 9 and the difficulty created by the absence of formulae for the dimensions of and multiplicities of weights in irreducible modules could be avoided.

We now give a survey of the methods used in this paper. Let (A,Y,V) statisfy the hypotheses of the main problem. We obtain preliminary information about the triple (A,Y,V) via induction. Choose a maximal parabolic P_A of A, with unipotent radical Q_A and Levi factor L_A. By the Borel–Tits theorem [2], there exists a parabolic subgroup P_Y of Y with $P_A \leq P_Y$ and $Q_A \leq Q_Y = R_u(P_Y)$. If L_Y is a Levi factor of P_Y, a result of Smith ([13]) implies that L_A' and L_Y' act irreducibly on the fixed point space V_{Q_A}. Hence, considering the projection of L_A' into the quasisimple components of L_Y' which act nontrivially on V_{Q_A}, we obtain a smaller rank version of the original problem. Theorem (7.1) of [12] is a complete solution of the main problem in the case where rankA = 1. Working inductively, we may describe V_{Q_A} (so partially describe V) and partially describe the embedding of L_A' in L_Y'. Though we are inducting on the rank of A, we handle the case where rank(A) = 2 and $Y = E_n$ in Chapters 6 – 9. Hence, in Chapters 4 and 5, we assume the results of the later chapters.

In Chapter 2, we establish machinery for studying general parabolic embeddings. As well, we prove results applicable only in the context of irreducibility on some module. (Several of the results are proven in [12].) Through this work, we can see the influence of the inductive information on (1) the projections of L_A' in the components of L_Y' which act trivially on V_{Q_A} and (2) the embedding of Q_A in Q_Y. Our considerations are as follows. With L_Y acting on Q_Y via conjugation, certain quotients of Q_Y may be regarded as modules for L_Y', and hence as modules for L_A'. We consider the image of the L_A' module $Q_A/[Q_A,Q_A]$ in these quotients. Of course, in an arbitrary parabolic embedding, Q_A may appear in few L_Y'

composition factors of Q_Y. But another consequence of Smith's result is the equality of the commutator subspaces $[V,Q_A]$ and $[V,Q_Y]$. The existence of particular weight spaces in $[V,Q_Y]$ often forces $Q_A/[Q_A,Q_A]$ to appear in particular quotients of Q_Y. Moreover, in most cases $Q_A/[Q_A,Q_A]$ must appear as an L_A' submodule. This will place restrictions on the projection of L_A' in the quasisimple components of L_Y' which act nontrivially on particular composition factors of Q_Y. We compare this with the inductively given information and perhaps produce a contradiction, or at least broaden our knowledge of the embedding $P_A \leq P_Y$.

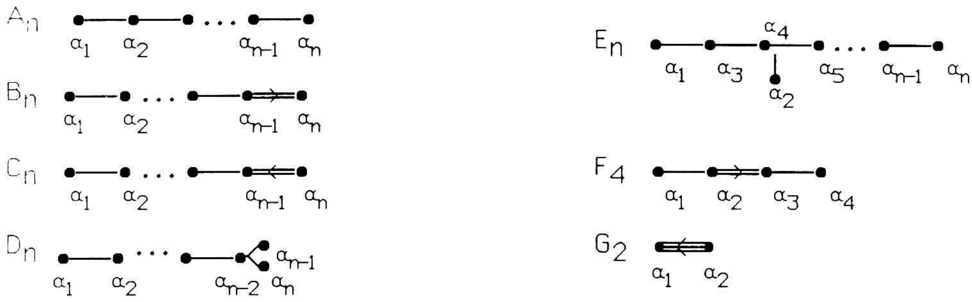
Throughout the paper, various numerical methods are employed as well. Since $[[V,Q_A],Q_A] \leq [V,Q_Y],Q_Y]$ and $[V,Q_A] = [V,Q_Y]$, $\dim([V,Q_Y]/[[V,Q_Y],Q_Y]) \leq \dim([V,Q_A]/[[V,Q_A],Q_A])$. Moreover, if $Z(L_A)^\circ \leq Z(L_Y)^\circ$ (which is usually implied by a suitable choice of P_Y), then the dimension of a $Z(L_Y)^\circ$ weight space of $[V,Q_Y]/[[V,Q_Y],Q_Y]$ is bounded by the dimensions of $Z(L_A)^\circ$ weight spaces of $[V,Q_A]/[[V,Q_A],Q_A]$. Seitz gives an explicit upper bound on the dimensions of the latter. This yields further restrictions on the high weight of $V|Y$. When the high weights of $V|A$ and $V|Y$ are almost explicitly determined, we attempt to show that $\dim V|Y$ exceeds the upper bound on $\dim V|A$ given by the Weyl degree formula. For this purpose, various methods for obtaining lower bounds on dimensions of kY modules are discussed in Chapter 1.

The absence of a "natural" module for the exceptional group Y gives rise to (expected) differences between Seitz's work in [12] and our work here. If Y is a classical group with natural (classical) module W, Seitz proves that in most cases, $W|A$ is irreducible and tensor indecomposable. This provides information about the restriction of elements of $\Sigma(Y)$ to a maximal torus of A and, coupled with an inductive hypothesis, usually implies that $V|A$ is a conjugate of a restricted module. As mentioned before, the tensor indecomposable situation is much easier to handle.

In our situation, where Y is an exceptional group, we may think of the "natural" module W as a restricted rational kY module of minimal dimension. However, there is no complete theory relating the subgroup structure of Y to its action on W. We do however use the module W whenever possible. We consider the action of L_A' on W, in particular the L_A' composition series of W. (This can be determined only when we have a fairly complete knowledge about the image of L_A' in L_Y') If dim(W) is relatively small (e.g., 26, 27 or 56) we can list all rational kA modules of this dimension, determine their L_A' composition series and compare with the given L_A' composition series of $W|L_A'$. Though fruitful in specific situations, this analysis does not serve the purpose that the natural module does for the classical groups. Rather, the bounded rank of the exceptional groups and our extensions of Seitz's results on parabolic embeddings enable us to restrict to the few possibilities of the Main Theorem.

For the convenience of the reader, many of the preliminary results from [12] are listed in this paper. It is useful to see that some of the results in Chapter 2 are natural extensions of the results on parabolic embeddings in [12]. A few essential theorems from [12], which we do not state, are often referenced. Theorem (7.1), mentioned already, is the solution of the main problem in case rank(A) = 1. Theorem (4.1) is a solution for the case where rank A = rank Y. (See Chapter 3 for a partial statement.) Theorem (8.1) gives the solution of the main problem for certain natural embeddings of classical groups. And finally, we refer to the list of all triples (A,Y,V), where Y has classical type, as the Main Theorem of [12].

Throughout the paper, we use the following labelling of Dynkin diagrams.



Let us make a few remarks about the notation in Table 1. The second column indicates the types of the groups A and Y, respectively. When the symbol " $\rightarrow$' " occurs, $A \leq B < Y$, for a closed, connected subgroup B, which is a commuting product of quasisimple groups as indicated. The notation means that either A projects surjectively to each of the simple factors of Y or some factor is of type B_2 and the projection is an A_1 acting irreducibly on the spin module for B_2. Moreover, in order to make sure $V|A$ is irreducible, it may be necessary for the projections to involve distinct field twists.

The third column describes the action of a covering group of A on a particular irreducible kY module, W. If Y is classical, W is the natural module for Y; if Y has type G_2, F_4, or E_6 (E_7 and E_8 do not arise), W is a restricted module of dimension 7 (6 if p=2), 26 (25 if p=3) or 27, respectively.

In the fourth and fifth columns the actions of A and Y on the module V are described, and in the last column any prime restrictions are indicated. Column 1 associates with each example a number. In the cases where there is an analogous zero characteristic example, Dynkin's numbering has been used. So I_1–I_{12}, II_1–II_9, III_1, IV_1–IV_{10}, V_1 and VI_1–VI_3 appear in [7]. Notation such as VI_1' refers to a variant of

Dynkin's VI_1. Examples MR_i are those where rankA = rankY and examples labelled S_1–S_9 are special examples occuring only when p = 2 or 3; these were found by Seitz in [12]. Examples T_1 and T_2 are found in this paper.

In conclusion, the author would like to express thanks to Gary Seitz, who suggested the problem, read an earlier version of this paper and offered useful advice throughout. As well, special thanks are given to Mark Reeder for numerous mathematical insights.

CHAPTER 1: PRELIMINARY LEMMAS

Let V be a finite dimensional vector space over an algebraically closed field k of characteristic $p > 0$, and let X be a semisimple, closed, connected subgroup of SL(V) with fixed maximal torus T. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a base for the root system $\Sigma(X)$ and let e_{α_i} and f_{α_i} denote the corresponding elements of the Lie algebra L(X). Labelling Dynkin diagrams as in Table 1, let λ_i be the fundamental dominant weight corresponding to α_i. Assume $V|X$ is irreducible and let λ be the high weight of V. Then $\langle\lambda,\alpha_i\rangle \geq 0$, for each i and V is said to be *restricted* if $\langle\lambda,\alpha_i\rangle < p$, for $1 \leq i \leq n$. For a subgroup $N < X$, let V_N denote the space of fixed points of N on V and [V,N] the commutator subspace $\langle v-nv \mid v \in V, n \in N\rangle$.

<u>(1.1).</u> (i) $V = V_1{}^{q_1} \otimes \cdots \otimes V_k{}^{q_k}$, where each V_i is an irreducible restricted module for X and $q_1, \ldots, q_k$ are distinct powers of p.

(ii) If V is restricted, then V is also irreducible when viewed as a module for L(X).

<u>Proof</u>: (i) is Steinberg's tensor product theorem (see [15]). For (ii) see Section A of [1].□

<u>(1.2).</u> ([13]) Let P be a proper parabolic subgroup of X with unipotent radical Q and Levi factor L. Then $L \cong P/Q$ acts irreducibly on V_Q.□

<u>(1.3).</u> ((1.7) of [12]) Let P be a proper parabolic subgroup of X with unipotent radical Q and Levi factor L. Then V/[V,Q] is irreducible for L. In fact, this quotient is L-isomorphic to $((V^*)_Q)^*$.□

<u>(1.4).</u> ([2]) Let $X \leq Y$, where Y is a closed, connected subgroup of SL(V) and let P be a parabolic subgroup of X with unipotent radical Q. There is a parabolic subgroup P_Y of Y with unipotent radical Q_Y such that

$P \le P_Y$ and $Q \le Q_Y$.□

(1.5). Let X, Y, and P be as in (1.4), and choose P_Y as in (1.4) minimal such that $P \le P_Y$ and $Q \le Q_Y = R_u(P_Y)$. Suppose $L_Y' = L_1 \cdots L_r$, where L_i is a simple normal subgroup of L_Y, with root system of classical type for $1 \le i \le r$. Then, $Z(L)^\circ \le Z(L_Y)^\circ$.

Proof: This follows from the proof of (2.8) in [12].□

(1.6). ((1.4) in [12]) Let $X \le Y$, $P \le P_Y$, $Q \le Q_Y$ be as in (1.4). Then, $V_Q = V_{Q_Y}$. So L and L_Y are reductive groups both acting irreducibly on $M = V_Q$ and the image of L in SL(M) is contained in the image of L_Y.□

(1.7). ((1.6) of [12]) Suppose X is simple. Then V can be expressed as the tensor product, $V = V_1 \otimes V_2$, of two nontrivial restricted kX-modules if and only if V is restricted and the following conditions hold:

(i) X has type B_n, C_n, F_4, or G_2, with p = 2,2,2,3, respectively.

(ii) V_1, V_2 may be arranged such that each V_i has high weight λ_i, $\lambda = \lambda_1 + \lambda_2$, and λ_1 (respectively, λ_2) has support on those fundamental dominant weights corresponding to short (long) fundamental roots.□

(1.8) Definition: Suppose X is simple.

(A) We say V is *basic* (respectively, *p-basic*) if the following conditions hold.

(i) V is restricted.

(ii) If X has type B_n, C_n, F_4, or G_2 with p = 2,2,2,3, respectively, then λ has short (respectively, long) support.

(B) If X and p are as in (ii), we say the pair (X,p) is *special.*

(1.9). Let X, P, Y, and V be as in (1.4) and suppose V|X is basic. Also, if (X,p) is special (respectively, $(G_2,2)$), assume $\Pi(X) - \Pi(L)$ is $\{\alpha\}$ (respectively, long).Then there exists a parabolic subgroup P_Y, of Y, such that the following hold:

(i) $P \le P_Y$ and $Q \le Q_Y = R_u(P_Y)$.

(ii) $L \le C_Y(Z(L)^\circ) \le L_Y$, a Levi complement to Q_Y in P_Y.

(iii) If T_Y is any maximal torus of Y containing T, then $T_Y \le L_Y$.

Proof: This follows from the first two paragraphs of the proof of (2.8) in [12].□

(1.10). ((2.16) of [12]) Let Y be a simple algebraic group and φ: Y $\longrightarrow$ SL(M) a basic representation. Suppose X is a simple, closed subgroup of Y and $\varphi|X$ is an algebraic conjugate of a restricted representation of X. Then $\varphi|X$ is restricted.□

(1.11). (i) V^* is irreducible with high weight $-w_0\lambda$, where w_0 is the long word in the fundamental reflections generating the Weyl group of X.

(ii) X leaves invariant a nondegenerate bilinear from on V if and only if $\lambda = -w_0\lambda$.

(iii) If X has type B_n, C_n, D_n for n even, E_7, E_8, F_4, or G_2, then X necessarily stabilizes a nondegenerate bilinear from on V.

(iv) If X has type A_n, D_n for n odd, or E_6, then X stabilizes a nondegenerate bilinear form on V if and only if $\lambda = \tau\lambda$, where τ is the graph automorphism of the Dynkin diagram of $\Sigma(X)$.

Proof: See Section 31 of [10].□

(1.12). ((1.14) of [12]) Let $X = SL_{n+1}$. For any integer p>c>0, the irreducible module V having high weight $c\lambda_1$ or $c\lambda_n$ is isomorphic to the space of homogeneous poylnomials of degree c in a basis of the usual module for X, or its dual. Thus, $\dim V = (1/n!)(c+1)(c+2)\cdots(c+n)$.□

(1.13). ((1.13) of [12]) Suppose $X = SL_2$. Then the weight spaces of T on V are of dimension 1.□

(1.14). Suppose $X = SL_2$. Let $\Pi(X) = \{\alpha\}$ and let λ_α be the fundamental dominant weight corresponding to α. Then X fixes a symplectic (respectively, orthogonal) form on the restricted, irreducible kX-module with high weight $n\lambda_\alpha$, where n is odd (even).

Proof: This follows immediately from Lemma 79 of [14].□

(1.15). Suppose $X = SL_2$. Let $\Pi(X) = \{\alpha\}$ and λ_α the fundamental dominant weight corresponding to α. Let W be the rational kX-module