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Duality and perturbation methods in critical point theory

PREFACE

The aim of these notes is to give a self-contained presentation of the min-max approach to critical point theory while emphasizing the role of *duality and perturbation* methods. Actually, this monograph originated in a project where we set out to show that *duality* is a fundamental concept that underlies many aspects of critical point theory. The goal was to try to reprove and improve selected results in min-max theory by exploiting the notion of *dual* families of sets and its ramifications. It turned out that, by adopting this point of view, the whole theory can be nicely developed and vastly enriched.

On the other hand, by *perturbation methods*, we mean the aspect of infinite dimensional critical point theory where, in order to deal with the possible lack of compactness or with the presence of degeneracy, one tries to modify the functional or the problem under study to a neighboring one that can be more manageable. We shall adhere to this methodology throughout these notes.

This monograph owes its existence to a very dear friend, Ivar Ekeland, who introduced me to non-linear analysis and, more importantly, influenced greatly my global vision of mathematics. Special thanks go to another dear friend, Bernard Maurey, for all the years of collaboration and from whom I learned so much. Many of the relevant examples included here are due to Gabriella Tarantello. I am very grateful to her for permitting me to include her results, some of which have not yet appeared in print. Many thanks to my graduate students, Guangcai Fang and David Robinson for their important contributions to the mathematical content as well as to the pedagogical aspect of this monograph. I would like to thank Rita McIlwaine from the staff of the department of mathematics at the University of British Columbia for helping me TEX some of this! Many thanks go to the senior editor of Cambridge University Press, David Tranah who makes book publishing look so easy! My deep gratitude goes to my wife Louise for coping with all that and much more . . .

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INTRODUCTION

Eigenvectors, geodesics, minimal surfaces, harmonic maps, conformal metrics with prescribed curvature, subharmonics of Hamiltonian systems, solutions of semilinear elliptic partial differential equations and Yang-Mills fields are all critical points of some functional on an appropriate manifold. This is not surprising since many of the laws of mathematics and physics can be formulated in terms of *extremum principles.*

Finding such points by minimization is as ancient as the least action principle of Fermat and Maupertuis, and the calculus of variations has been an active field of mathematics for almost three centuries. For more general, unstable extrema, the methods have a more recent history. Two, not unrelated, theories are available for dealing with the existence of such points: *Morse theory* and *the min-max* methods (or the calculus of variations in the large) introduced by G. Birkhoff and later developed by Ljusternik and Schnirelmann in the first half of this century. Currently, both theories are being actively refined and extended in order to overcome the limitations to their applicability in the theory of partial differential equations: limitations induced by the infinite dimensional nature of the problems and by the prohibitive regularity and non-degeneracy conditions that are not satisfied by present-day variational problems.

In this monograph, we will be concerned with the old problems of existence, location, multiplicity and structure of critical points via the methods of the calculus of variations in the large, but we shall present a new point of view that might help in dealing with some of the difficulties mentioned above. Actually, we shall build upon the well known min-max methods that are presently used in non-linear differential equations and which are well documented in the lecture notes of Rabinowitz [R 1] and the recent books of Ekeland [Ek 2], Mawhin-Willem [M-W] and Struwe [St]. However, unlike the above mentioned monographs, our emphasis will be on trying to develop, in a systematic way, a "general theory" for variationally generated critical points, that can be applied in a variety of situations, under a minimal set of hypotheses. But our real goal is to advertise a whole array of newly discovered *duality and perturbation methods* that are instrumental in carrying out the refinements of that theory.

One of the central questions we address is the following: what can one say about the *critical structure* of a functional defined on an infinite dimensional domain without imposing the usual compactness conditions *à la Palais-Smale*, or the non-degeneracy conditions *à la Fredholm*? We shall present various new variational principles which, besides ensuring the existence of critical points under minimal hypotheses, give valuable information about their location, their multiplicity and their structure. These principles will yield most of the classical results. However we shall emphasize those applications where the standard methods do not apply but the ones offered in this monograph do.

Here is an overview of our approach. Consider a C^1-functional φ on a smooth infinite dimensional manifold X. If φ is bounded below on X, then one can try to find a critical point at the level $c = \inf_X \varphi$ by checking whether the infimum is attained. If φ is indefinite or if one is looking for *unstable* critical points, one then considers a family $\mathcal{F}$ of compact subsets of X that is stable under a certain class of homotopies and then shows that φ has a critical point at the level

$$c = c(\varphi, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x).$$

In both cases, one can easily find an *almost critical sequence* at the level c; that is a sequence $(x_n)_n$ in X satisfying:

$$\lim_n \varphi(x_n) = c \quad \text{and} \quad \lim_n \|d\varphi(x_n)\| = 0. \tag{1}$$

The main problem of existence reduces to proving that such a sequence is convergent. This is usually where the hard analysis is needed. Any function that possesses such a property is said to satisfy the *Palais-Smale condition at the level c* (in short, $(PS)_c$). However, various interesting functionals originating in partial differential equations and in differential geometry do not satisfy such a property, or they may only satisfy it for certain levels c. These problems usually occur in situations involving the critical exponent in the Sobolev embedding theorems, or in the cases where *scale or gauge invariance* requirements give rise to non-compact group actions.

Our main purpose in this direction is to try to find almost critical sequences for φ that possess some extra properties which might help to ensure their convergence. Here is the first idea that comes to mind: one can find an almost critical sequence that is arbitrarily close to any *min-maxing sequence* $(A_n)_n$ *in* $\mathcal{F}$. That is, if $(A_n)_n$ in $\mathcal{F}$ is such that $\lim_n \max_{A_n} \varphi = c$, then one can find a sequence $(x_n)_n$ that satisfies (1) as well as

$$\lim_n \operatorname{dist}(x_n, A_n) = 0. \tag{2}$$

What is needed then, is *a Palais-Smale condition along one min-maxing sequence* $(A_n)_n$ *in* $\mathcal{F}$. As we shall see in Chapter 3, this weakening of the Palais-Smale condition turns out to be relevant for the solution of a *resonance problem.*

Another point of view consists of considering a family $\mathcal{F}^*$ of closed sets that is *dual to* $\mathcal{F}$ (i.e., satisfying $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}^*$ and $A \in \mathcal{F}$) and such that

$$\sup_{F \in \mathcal{F}^*} \inf_{x \in F} \varphi(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x) = c.$$

One of the main results of this monograph (Theorem 4.5) shows that one can then construct an almost critical sequence that is also arbitrarily close to any *max-mining sequence* $(F_n)_n$ *in* $\mathcal{F}^*$. In other words, if $(F_n)_n$ in $\mathcal{F}^*$ is such that $\lim_n \inf_{F_n} \varphi = c$, then one can find a sequence $(x_n)_n$ that satisfies (1), (2) and

$$\lim_n \operatorname{dist}(x_n, F_n) = 0. \tag{3}$$

There are three important features to this new min-max principle. First, we could expect that the topology, which provides the dual families, can sometimes help the analysis involved in proving the convergence of such pseudo-critical sequences. We shall see in Chapter 8, that in various examples, one can indeed push back the threshold of non-compactness by ensuring that an almost critical sequence is arbitrarily close to a certain dual set.

Another aspect of that refinement is that it helps relax the boundary conditions: Indeed, the homotopies that preserve the elements of the family $\mathcal{F}$ are usually required to leave a certain boundary B invariant. In that case, the standard existence results require that $\sup \varphi(B) < c = c(\varphi, \mathcal{F})$. In Chapters 4 and 5, we exhibit situations where the boundary condition can be relaxed to $\sup \varphi(B) = c$, especially in the presence of dual sets.

The third and most important feature of this principle is that it locates *critical points* whereas the classical min-max principle only identifies possible *critical levels.* Indeed, as mentioned in (3) above, we could find, under adequate compactness conditions, a critical point on any dual set F. The largest dual set is clearly $\{\varphi \geq c\}$. The trick is to find proper subsets F of $\{\varphi \geq c\}$ that still intersect all the members in $\mathcal{F}$, since the smaller F is, the more information we have about the critical points it contains. By appropriate choices of dual sets, we manage to prove various old and new results concerning the multiplicity and the Morse indices of critical points generated by this procedure. In Chapter 6, we use this method to classify the critical points generated by the mountain pass theorem in a setting that is not covered by Morse theory. In Chapter 7, we show how this same information about the location of critical points on various dual sets gives easy proofs of the standard multiplicity results, while it leads naturally to new and unexpected ones. In Chapter 9, it helps us relate the Morse index of a critical point to the *topological dimension* (homotopic, cohomotopic or homological) of the family $\mathcal{F}$ used to obtain it in the case of non-degenerate C^2-functionals on Riemannian manifolds. In Chapter 10, we study the degenerate case and find that the same principle gives new types of multiplicity results: these concern the size of sets of critical points having a common estimate on their Morse indices.

Let us return to the possibility of "improving" the Palais-Smale sequences in order to ensure their convergence. Suppose now that φ is a C^2-functional. One can then try to get an almost critical sequence $(x_n)_n$ with some information about the second derivatives $d^2\varphi(x_n)$. For example, this can be done in minimization problems: one can then construct an almost critical sequence $(x_n)_n$ that is minimizing and that satisfies for each $n \in \mathbb{N}$:

$$\langle d^2\varphi(x_n)v, v\rangle \geq -\|v\|^2/n \quad \text{for any } v \in X. \tag{4}$$

One way to obtain such sequences consists of establishing "perturbed variational principles". (A typical example would be Ekeland's Theorem). If, due to the lack of compactness, we cannot find critical points for a given function, can one then perturb it so that the new functional has critical points of the kind that is expected for the original one. This type of result is much stronger than finding almost critical points for the original function. Indeed, if the perturbation is C^2-small, then knowing that the almost critical point is a true critical point for the new functional allows us to use Morse theory and therefore to get some information about the Hessian of the original function at that point.

For minimization problems, several results of this kind have recently been established with various degrees of generality. In Chapter 1, we include a general variational principle where the perturbations can be taken to be as smooth as the norm on the domain of the functional under study. As an illustration, we give a result of P. L. Lions about the solutions of the *Hartree-Fock equations* as well as an application to the existence of *viscosity solutions* for first-order *Hamilton-Jacobi equations.* We also give, in Chapter 2, two recent results of a similar nature: In the first, which can be applied in reflexive Banach spaces, the perturbations can be taken to be linear while the second covers spaces as "topologically and geometrically bad" as L^1, provided one settles for plurisubharmonic perturbations. This will be used in Chapter 2 to get generic minimization results in the case of critical exponents and non-zero data.

Perturbed variational principles for problems not involving minimization are more involved and one should expect to need *hyperbolic perturbations.* Results of this type are very recent and are not yet in their final form. However, we shall present in Chapter 11, a new method devised by Fang and Ghoussoub, for constructing directly –without establishing the perturbation result– an almost critical sequence with the appropriate second order information, provided one has an additional assumption of Hölder continuity on the first and second derivatives. For example, in the context of *the mountain pass theorem,* we obtain an almost critical sequence $(x_n)_n$ and a sequence of subspaces $(E_n)_n$ of codimension one such that the sequence $(x_n)_n$ satisfies, in addition to (1), (2) and (3) above, the following condition:

$$\langle d^2\varphi(x_n)v, v\rangle \geq -\|v\|^2/n \quad \text{for any } v \in E_n. \tag{5}$$

In other words, for each $n \in \mathbb{N}$, $d^2\varphi(x_n)$ has *at most one* eigenvalue below $-1/n$, which clearly implies that any potential cluster point for $(x_n)_n$ will be a critical point of Morse index at most one. More importantly, and as already noted by P. L. Lions in his study of the Hartree-Fock equations, this additional information is sometimes crucial for proving the convergence of such sequences, especially when the standard Palais-Smale condition is not satisfied. We shall include this example as an application of our results in the homotopic case.

Let us mention that another weakening of the Palais-Smale condition also turned out to be relevant in some geometric problems. It is the (PS) condition along a fixed orbit of the *negative gradient flow* of the functional: that is when compactness holds for those pseudo-critical sequences of the form

$$x_n = \sigma(t_n, x) \tag{6}$$

where σ is the solution of the Cauchy problem

$$\dot{\sigma} = -\nabla\varphi(\sigma), \quad \sigma(0, x) = x.$$

We shall not discuss this phenomenon here and we refer the reader to the book of Bahri [Ba 2] for a detailed analysis of several important examples.

Throughout this monograph, we have also emphasized the role of various *local perturbation methods* that are now available and which are interesting in their own right. These techniques consist of changing a problem appropriately to a neighboring one that can be more manageable. Here is a sample: In Theorem 4.5, we perturb the function and the variational setting to reduce the new refined min-max principle to the old one. In Theorem 5.2, we use yet another perturbation to reduce the new relaxed boundary condition to the classical one. In chapter 10, we use the Marino-Prodi perturbation method to change a degenerate case to a non-degenerate one. A method for restricting a homotopy-stable class of sets to submanifolds is implemented, while another kind of perturbation is needed to isolate certain subsets of critical points.

We have tried to make this monograph as self-contained as possible, especially for the part that deals with "the general theory". It was hard to do the same with the examples and therefore they may seem sketchy at times. However, the necessary results and concepts are included – some without proofs – in an appendix prepared skillfully by David Robinson. This is definitely not a comprehensive study of critical point theory, nor of any subset of the theory of partial differential equations. It is merely a collection of some recently formulated variational principles, followed by some examples justifying their relevance. Our main goal is to make these new methods accessible to non-linear analysts, hoping that more interesting applications will follow. To illustrate these principles, we chose the simplest examples known to us, and often did not include the strongest known versions, in order to avoid the complications that were irrelevant to our discussion.

Before closing this introduction, we should mention that at present, there are at least two other approaches that have proved very successful in dealing with the difficulties of infinite dimensional critical point theory and which are unfortunately missing from this monograph. We have in mind Conley's and Floer's approaches to Morse theory ([Co], [Fl 1,2]) and the machinery involving *critical points at infinity* that was mainly developed by A. Bahri [Ba 2]. The reason for these omissions is simply that both of these theories are substantialy more involved than the elementary approach adopted in this monograph since they require a much heavier backgroung from algebraic topology and geometry.

CONTENTS

1

LIPSCHITZ AND SMOOTH

PERTURBED MINIMIZATION PRINCIPLES

We start this chapter by stating and proving Ekeland's well known variational principle since it will be used frequently throughout this monograph. We also give some of its lesser known applications to constrained minimization problems that eventually yield *global* critical points for the functional in question. We introduce the *Palais-Smale condition around a set* and we present the first of many examples which show its relevance. We give an existence result for nonhomogeneous elliptic equations involving the critical Sobolev exponent, due to Tarantello. We then establish the more recent *smooth variational principle* of Borwein-Preiss and we apply it to the study of Hartree-Fock equations for Coulomb systems as was done by P.L. Lions. Finally, we follow the ideas of Ghoussoub and Maurey to deal with the more general problem of identifying those classes of functions that can serve as *perturbation spaces* in an appropriate minimization principle. As an application, we give a result of Deville et al, stating that the perturbations can be taken to be as smooth as the norm of the Banach space involved. We then apply this result to the problem of existence and uniqueness of *viscosity solutions* for first order Hamilton-Jacobi equations on general Banach spaces.

1.1 Ekeland's variational principle

The following theorem will be of constant use throughout this monograph. The applications of this principle to non-linear analysis are numerous and well documented in several books ([A-E], [Ek 1], [De]). We shall only be concerned with those that are relevant for this monograph.

Theorem 1.1 (Fig 1.1): *Let (X, d) be a complete metric space and* *consider a function $\varphi : X \to (-\infty, +\infty]$ that is lower semi-continuous, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ and $\lambda > 0$ be given and let $u \in X$ be such that $\varphi(u) \leq \inf_X \varphi + \varepsilon$. Then there exists $v_\varepsilon \in X$ such that*

(i) $\varphi(v_\varepsilon) \leq \varphi(u)$

(ii) $d(u, v_\varepsilon) \leq 1/\lambda$

(iii) For each $w \neq v_\varepsilon$ in X, $\varphi(w) > \varphi(v_\varepsilon) - \varepsilon\lambda d(v_\varepsilon, w)$.

Proof: We shall show the existence of $v_\varepsilon \in X$ such that

$$\varphi(v_\varepsilon) \leq \varphi(u) \tag{1}$$

$$d(u, v_\varepsilon) \leq 1 \tag{2}$$

and, for each $w \neq v_\varepsilon$ in X,

$$\varphi(w) > \varphi(v_\varepsilon) - \varepsilon d(v_\varepsilon, w). \tag{3}$$

The theorem will then follow by replacing the metric d by λd.

Introduce the following partial order on X

$$w \prec v \quad \text{provided } \varphi(w) + \varepsilon d(v, w) \leq \varphi(v).$$

We construct inductively a sequence (u_n) as follows: Start with $u_0 = u$. Once u_n is known, let $S_n = \{w \in X : w \prec u_n\}$ and choose $u_{n+1} \in S_n$ such that

$$\varphi(u_{n+1}) \leq \inf_{S_n} \varphi + \frac{1}{n+1}.$$

Clearly, $S_{n+1} \subset S_n$, as $u_{n+1} \prec u_n$, and since φ is lower semi-continuous, S_n is closed. Now, if $w \in S_{n+1}$, $w \prec u_{n+1} \prec u_n$ and hence

$$\varepsilon d(w, u_{n+1}) \leq \varphi(u_{n+1}) - \varphi(w) \leq \inf_{S_n} \varphi + \frac{1}{n+1} - \inf_{S_n} \varphi = \frac{1}{n+1}$$

so that $\operatorname{diam}(S_{n+1}) \leq \frac{2}{\varepsilon(n+1)}$. Since X is complete, this implies that

$$\bigcap_n S_n = \{v_\varepsilon\} \tag{4}$$

for some $v_\varepsilon \in X$. In particular, $v_\varepsilon \in S_0$, which means that

$$\varphi(v_\varepsilon) \leq \varphi(u) - \varepsilon d(u, v_\varepsilon) \leq \varphi(u)$$

and

$$d(u, v_\varepsilon) \leq \varepsilon^{-1}(\varphi(u) - \varphi(v_\varepsilon)) \leq \varepsilon^{-1}(\inf_X \varphi + \varepsilon - \inf_X \varphi) = 1.$$

To obtain (iii), it suffices to notice that if $w \prec v_\varepsilon$, then for each $n \in \mathbb{N}$, $w \prec u_n$ so that $w \in \cap_n S_n$ and, by (4), $w = v_\varepsilon$.

Our first application is to the existence of minimizing sequences that are also almost critical. Throughout this monograph and unless stated otherwise (as in sections 1.8 and 1.9), we shall say that a real-valued function φ on a Banach space X is *differentiable* if it is *Fréchet differentiable*; that is for any $x \in X$, there is $p \in X^*$ (denoted $\varphi'(x)$) such

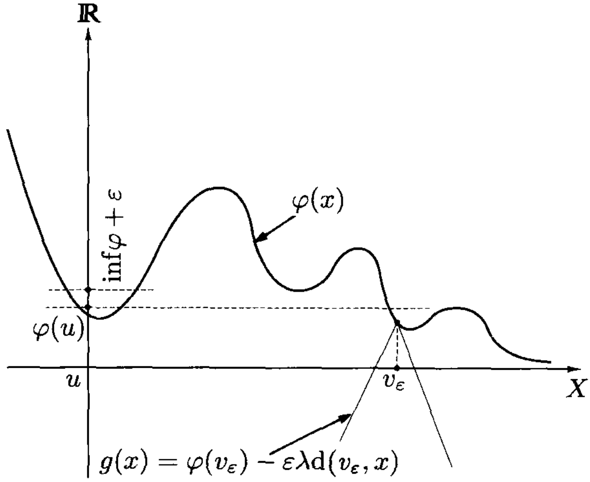

Fig. 1.1.

that

$$\lim_{t\to 0} t^{-1}[\varphi(x+th) - \varphi(x) - \langle p, th\rangle] = 0$$

uniformly for $h \in X$ with $\|h\| = 1$.

We shall say that φ is a C^1-functional if $x \to \varphi'(x)$ is a continuous map from X to its dual X^*.

Corollary 1.2: *Let X be a Banach space and let $\varphi : X \to \mathbb{R}$ be a function that is bounded from below and differentiable on X. Then, for each $\varepsilon > 0$ and for each $u \in X$ such that $\varphi(u) \leq \inf_X \varphi + \varepsilon^2$, there exists $v_\varepsilon \in X$ such that*
(i) $\varphi(v_\varepsilon) \leq \varphi(u)$.
(ii) $\|u - v_\varepsilon\| \leq \varepsilon$.
(iii) $\|\varphi'(v_\varepsilon)\| \leq \varepsilon$.

Proof: For $\varepsilon > 0$ given, use Theorem 1.1 with $\lambda = 1/\varepsilon$ to find $v_\varepsilon \in X$ such that (i), (ii) hold as well as

$$\varphi(w) > \varphi(v_\varepsilon) - \varepsilon\|v_\varepsilon - w\| \qquad (*)$$

for all $w \neq v_\varepsilon$ in X. By applying $(*)$ to $w = v_\varepsilon + th$ with $t > 0, h \in X, \|h\| = 1$, we get $\varphi(v_\varepsilon + th) - \varphi(v_\varepsilon) > -\varepsilon t$. Dividing both sides by t and letting $t \to 0$, we obtain $-\varepsilon \leq \langle \varphi'(v_\varepsilon), h\rangle$ for all $h \in X$ with $\|h\| = 1$, and hence (iii) is verified.

Corollary 1.3: *Let X be a Banach space and let $\varphi : X \to \mathbb{R}$ be a function that is bounded from below and differentiable on X. Then, for each minimizing sequence (u_k) for φ, there exists a minimizing sequence (v_k) for φ such that $\varphi(v_k) \leq \varphi(u_k)$, $\lim_k \|u_k - v_k\| = 0$ and $\lim_k \varphi'(v_k) = 0$.*

Proof: If (u_k) is a minimizing sequence for φ, take

$$\varepsilon_k = \begin{cases} \varphi(u_k) - \inf_X \varphi & \text{if } \varphi(u_k) - \inf_X \varphi > 0 \\ 1/k & \text{if } \varphi(u_k) - \inf_X \varphi = 0 \end{cases}$$

and then take v_k associated to u_k and ε_k in Corollary 1.2.

Definition 1.4: Say that φ verifies the *Palais-Smale condition at the level* c (in short $(PS)_c$), if any sequence $(x_n)_n$ satisfying $\lim_n \varphi(x_n) = c$ and $\lim_n \|\varphi'(x_n)\| = 0$ has a convergent subsequence.

Throughout this monograph, we shall denote by K_c the set of critical points at level c, i.e.,

$$K_c = \{x \in X; \varphi(x) = c, d\varphi(x) = 0\}.$$

The following two corollaries show that the (PS) condition is quite restrictive. In particular, it forces the function to be *coercive*, i.e., $\liminf_{\|u\|\to\infty} \varphi(u) = \infty$.

Corollary 1.5: *Let φ be C^1-function on a Banach space X.*

(i) If φ is bounded below and verifies $(PS)_c$ with $c = \inf_X \varphi$, then every minimizing sequence for φ is relatively compact. In particular, φ achieves its minimum at a point in K_c.

(ii) If $d = \liminf_{\|u\|\to\infty} \varphi(u)$ is finite , then φ does not verify $(PS)_d$.

Proof: (i) follows immediately from Corollary 1.3. For (ii), we shall show the existence of a sequence $(u_n)_n$ in X such that $\|u_n\| \to \infty$, $\varphi(u_n) \to d$ and $\|\varphi'(u_n)\| \to 0$.

For that, define for $r \geq 0$ the function

$$m(r) = \inf_{\|u\|\geq r} \varphi(u).$$

Clearly $m(r)$ is nondecreasing and $\lim_{r\to\infty} m(r) = d$. For $\varepsilon < 1/2$, find $r_0 \geq 1/\varepsilon$ such that $d - \varepsilon^2 \leq m(r)$ for $r \geq r_0$, then choose u_0 with $\|u_0\| \geq 2r_0$ such that

$$\varphi(u_0) < m(2r_0) + \varepsilon^2 \leq d + \varepsilon^2.$$

Apply now Ekeland's theorem in the region $D = \{\|u\| \geq r_0\}$, to find v_0 with $\|v_0\| \geq r_0$ such that

$$\varphi(v_0) \leq \varphi(u) - \varepsilon\|u - v_0\| \quad \text{for all } u \in D.$$

It follows that

$$d - \varepsilon^2 \leq m(r_0) \leq \varphi(v_0) \leq \varphi(u_0) - \varepsilon\|u_0 - v_0\|.$$

Hence $\|u_0 - v_0\| \leq 2\varepsilon$ and $\|v_0\| > r_0$. Since v_0 belongs to the interior of the region D, the argument in Corollary 1.2 gives that $\|\varphi'(v_0)\| \leq \varepsilon$.

Corollary 1.6: *Let φ be C^1-function satisfying the (PS) condition on a Banach space X. If u_0 is a local minimum for φ, then there is $\varepsilon > 0$ such that the following alternative holds:*

(i) Either $\varphi(u_0) < \inf\{\varphi(u) : \|u - u_0\| = \alpha\}$ for some $0 < \alpha < \varepsilon$,
(ii) or for each α with $0 < \alpha < \varepsilon$, φ has a local minimum at a point u_α with $\|u_\alpha - u_0\| = \alpha$ and $\varphi(u_\alpha) = \varphi(u_0)$.

Proof: Let $\varepsilon > 0$ be such that $\varphi(u_0) \leq \varphi(u)$ for $\|u - u_0\| \leq \varepsilon$. If (i) does not hold, then for any given α with $0 < \alpha < \varepsilon$, we have

$$\varphi(u_0) = \inf\{\varphi(u) : \|u - u_0\| = \alpha\} \tag{1}$$

Let $\delta > 0$ be such that $0 < \alpha - \delta < \alpha + \delta < \varepsilon$ and consider the restriction of φ to the ring $R = \{u \in X : \alpha - \delta \leq \|u - u_0\| \leq \alpha + \delta\}$. From (1), we can find a sequence $(u_n)_n$ in R such that

$$\|u_n - u_0\| = \alpha \text{ and } \varphi(u_n) \leq \varphi(u_0). \tag{2}$$

By Ekeland's theorem, we can then find $(v_n)_n$ in R such that

$$\varphi(v_n) \leq \varphi(u_n), \qquad \|u_n - v_n\| \leq \frac{1}{n} \tag{3}$$

and

$$\varphi(v_n) \leq \varphi(u) + \frac{1}{n}\|u - u_n\| \text{ for all } u \in R. \tag{4}$$

If n is large enough, we get from (3) that v_n is in the interior of R, which then implies that $\|\varphi'(v_n)\| \leq \frac{1}{n}$. The (PS) condition now ensures that a subsequence of the $(v_n)_n$ converges to a point u_α. It is clear that φ has a local minimum at u_α and that $\|u_\alpha - u_0\| = \alpha$ while $\varphi(u_\alpha) = \varphi(u_0)$.

Remark 1.7: If u_0 is a strict local minimum, then clearly alternative (ii) cannot hold, and we then obtain an $\alpha > 0$ such that

$$\varphi(u_0) < \inf\{\varphi(u) : \|u - u_0\| = \alpha\}.$$

1.2 Constrained minimization and global critical points

The (PS) condition being so restrictive, one can try to find pseudo-critical sequences with some additional properties that might help in proving their convergence. To do that, we introduce the following concept which will play a central role throughout this monograph.

Definition 1.8: Say that φ verifies the *Palais-Smale condition at the level c and around the set F* (in short, $(PS)_{F,c}$) if any sequence $(x_n)_n$ in X verifying $\lim_n \varphi(x_n) = c$, $\lim_n \|\varphi'(x_n)\| = 0$ and $\lim_n \operatorname{dist}(x_n, F) = 0$ has a convergent subsequence.

The most naive approach for finding such points consists of minimizing the functional φ on the submanifold F and to check whether such a relative minimum is a global critical point for φ. We have observed that this is the case if, for instance, the points obtained via Ekeland's theorem are in the interior of the constraint set. In the sequel, we shall present other settings where this method can apply. On the other hand, in Chapter 4, we shall implement a global (unconstrained) variational