A Focus on Fractions

Bringing Research to the Classroom

Marjorie M. Petit, Robert E. Laird, and **Edwin L. Marsden**

ROUTLEDGE

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Dedication

This book is dedicated to the teachers and students of Vermont and Alabama who participated in Vermont Mathematics Partnership's Ongoing Assessment Project studies and scale-up and to the mathematics education researchers upon whose work this book is built. Without the foundational research by mathematics education researchers and the hundreds of interactions with Vermont and Alabama educators, this book would not have been possible.

Preface

It is safe to assume that you may have picked up this book because you are an educator who is baffled about why many students (elementary, middle school, high school, or college) have profound difficulties learning and applying fraction concepts. Be assured that you are not alone. Fractions are considered by many to be among the most difficult topics in the elementary school curriculum. As a matter of fact, in a recent national report, mathematicians and mathematics educators alike reported that problems with learning fractions interfere with learning other mathematics topics and continue to plague adults in daily tasks.

> Difficulty with learning fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra. It has also been linked to difficulties in adulthood, such as failure to understand medication regimens.
>
> (National Mathematics Panel Report, 2008)

A student solution like the one found in Figure i.1 underscores this point. Without an understanding of mathematics education research, one is left to wonder how it is possible that a student who can accurately add fractions can have little understanding about the magnitude of $\frac{23}{24}$.

Figure i.1

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20
B. 8
C. $\frac{1}{2}$
D. 1

Explain your answer.

$\frac{1}{12} + \frac{7}{8} = \frac{2}{24} + \frac{21}{24} = \frac{23}{24}$ is closest to 20.

This student work and all subsequent questions and student work are from VMP OGAP materials funded by the US Department of Education (Award Number S366A020002) and the National Science Foundation (Award Number EHR-0227057)

A Focus on Fractions: Bringing Research to the Classroom is designed to communicate important mathematics education research about how students develop an understanding of fractions concepts, common errors that students make, and preconceptions or misconceptions that may interfere with students learning new concepts and solving problems involving fractions. Educators have found that knowledge of these aspects of student learning related to fractions has a powerful impact on their teaching and on their students' learning.

This book grew out of a successful formative assessment project (VMP OGAP[1]) that is based on mathematics education research related to the teaching and learning of fractions. The student work samples, teacher advice, and VMP OGAP data referenced throughout the book came from interactions with hundreds of teachers and thousands of students in Vermont, Alabama, and Michigan between 2004 and 2009. Because of this, the ideas within these pages have been fostered, influenced, and practiced by countless educators in real educational settings.

Teachers in these VMP OGAP studies report that the knowledge contained in this book has helped them to:

- better understand evidence in student work;
- use the evidence to inform instruction;
- strengthen first wave instruction;
- understand the purpose of activities in their mathematics program, maximizing the potential of their instruction materials; and
- better understand fraction concepts.

The comment below is typical of how most teachers react once they gain an understanding of important mathematics education research related to how students learn fraction concepts.

> Before I used these materials I had no idea what it was about fractions that my students did not understand. It would not have occurred to me, for example, that students had a difficult time understanding that proper fractions were less than one, or that some students thought a fraction was less than zero. It was surprising to find out that one of the biggest problems for elementary students as they learn fractions is that they see proper fractions as two whole numbers and not a single value. More alarming was understanding that my instruction may have been reinforcing this idea.
>
> (VMP OGAP, personal communication, 2005)

A Book Designed for Classroom Teachers, In-service, and Pre-service Training

From its earliest inception, this has been a book for classroom teachers. Teachers will recognize that the vignettes used in various chapters describe real

[1] Vermont Mathematics Partnership Ongoing Assessment Project (http://vermontinstitutes.org/index.php/vmp/ogap)

issues that teachers face as they contemplate how best to teach fraction concepts to their students. The authentic student work used throughout the book, as well as the questions at the end of each chapter, provide teachers with numerous opportunities to analyze student thinking and to consider instructional strategies for their own students. Answers to the questions can be found at www.routledge.com/9780415801515. Each chapter also provides an opportunity for teachers to link concepts from the chapter to their own instructional materials/programs. Teachers have found these instructional links to be vital in helping to understand how their math program addresses the related research findings and in helping them to plan an effective unit of instruction.

Educators providing in-service training will also find *A Focus on Fractions: Bringing Research to the Classroom* to be a valuable component in helping their teachers become more successful teachers of fractions. In fact, several of the vignettes, the instructional links, most of the important ideas in this book, and a majority of the questions at the end of each chapter were first piloted through in-service courses. In-service leaders and participants alike found that the ideas, educational research, student work, instructional links, and end of chapter questions contained in this book provided a platform for meaningful exploration of substantive aspects of teaching fractions. Groups of math teachers from the same school as well as grade level teaching teams have found that working together around these ideas is particularly powerful.

Instructors working with pre-service teachers will find the numerous samples of student work to be valuable in bringing authentic student thinking into class discussions. In addition, pre-service teachers will be introduced to important educational research related to fractions and provided with many opportunities to "see" the research in student work, discuss research with peers, and consider the important instructional decisions that are at the heart of being an effective teacher of mathematics.

Bridging the Gap between Researchers and Practitioners

We have found that people outside the education profession are often surprised that, in general, educators are not aware of mathematics education research. In turn, teachers are surprised by two things. First, they are astonished that this research exists and that they were never exposed to it. Second, they are amazed at how readily the evidence in their students' work conforms to the findings in the research. These materials have been designed to help bridge the gap between what mathematics education researchers have discovered about the learning of fraction concepts and what teachers need to know to make effective instructional decisions.

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VMP OGAP Design Team Members:

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OGAP National Advisory Board:

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- Edward Silver, University of Michigan
- Judith Zawojewski, Illinois Institute of Technology.

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Contents

1
Modeling and Developing Understanding of Fractions

Big Ideas

- The use of models should permeate instruction, not be just an incidental experience, but a way of thinking, solving problems, and developing fraction concepts.
- Students should interact with a variety of models that differ in perceptual features.
- Modeling is a means to the mathematics, not the end.

Most teachers understand that models should be a part of fraction instruction. However, they often have many questions about their use, such as:

- What is the purpose of using models? To build fraction concepts? To use as a tool for solving problems? Other?
- My math program only uses one type of model (e.g., circle model). Is that OK?
- What is the best way to use models? For example, I am uncertain when my students should use the fraction strips they made at the beginning of the unit. Any time they want? Only for certain activities?
- Why can my students shade $\frac{3}{4}$ of a figure using an area model one day, and the very next day not be able to locate $\frac{3}{4}$ on a number line, or find $\frac{3}{4}$ of a set of objects? It just doesn't make sense to me.
- I have sixth grade students who use models to compare fractions. Is that OK? How can I move them from using a model to more efficient strategies (e.g., number sense, common denominators)?
- My textbook never provides students with an opportunity to make their own models? Is that OK?

This chapter *begins* to address some of these and other questions/issues as teachers are making instructional use of models. Subsequent chapters will show how modeling helps to build specific mathematics concepts (e.g., operations with fractions, equivalence).

The vignette that follows as well as the message that "models are the means to the mathematics, not the end" set the stage for understanding the importance of using models to help students to build an understanding of fraction concepts. While addressing some of the teacher questions about using models, it also illustrates the fine balance in using models to develop understandings without developing an over-reliance on models.

A Case Study—When Models are Used like Calculators

Mr. Smith is a fourth grade teacher who has been using the same mathematics program for the past five years. The program teaches fraction concepts through the use of only one model—the circle model. As a part of the instruction guided by this program, students make circle models representing halves, thirds, fourths, fifths, sixths, sevenths, . . ., fourteenths, which are put on display and used in all aspects of the unit. Mr. Smith has always been comfortable with using just circle models for fraction instruction.

This past year Mr. Smith participated in the Ongoing Assessment Project (OGAP) Study. He noticed that the OGAP questions did not always use the circle model, but included a variety of area models, number lines, set models, and models involving manipulatives such as pattern blocks and geoboards. However, since he was familiar with using the circle model, he charged ahead.

Midway through the unit he gave the students a question that involved comparing $\frac{3}{7}$ and $\frac{7}{8}$. The students asked if they could use their circle models on display to answer the question. Mr. Smith said they could if they needed to, but was hoping that they would not feel the need to use them.

Mr. Smith was very disappointed with what happened and was beginning to question the decision to just use circle models. With the exception of three students, all the students felt that they could **not** compare the fractions without the use of the models on the wall. He was hoping that his students would be able to visualize and justify $\frac{7}{8}$ as greater than $\frac{3}{7}$ using student drawn models or justifications based on $\frac{1}{2}$ as a benchmark. He thought that this was an easy comparison. However, instead of the models helping his students to internalize (generalize) the ideas behind the concepts, he realized that his students were using the pre-made circle models as the only way to compare fractions in the same way that students sometimes inappropriately use calculators as the only way to make calculations.

It may be that Mr. Smith's reliance on one type of model limited his students' abilities to make the important conceptual leap he intended. He was not sure. Mr. Smith realized that he needed to learn more about how to use models in his instruction and why using different models could help his students to internalize and generalize the mathematical ideas.

This vignette paints a picture of a classroom in which only one type of model (fraction circles) was used, and a classroom in which students relied on the models they made at the beginning of the unit as if they were reaching for a calculator to do a simple calculation. It may be possible that the students could compare these fractions without the pre-made models, but it was becoming clear to Mr. Smith that his students' use of models was not necessarily helping them to internalize fraction concepts in the way that he intended.

According to research Mr. Smith inadvertently made two mistakes in his use of fractions circles that may have led to his students not internalizing the concepts he intended to develop.

- His students used the fraction circles in a *"rote" way, not tied to the mathematical ideas that are embodied in the fraction circles* (Clements, 1999). This led to their dependence on the circles to compare fractions.
- He used only one model, while research suggests *that learning is facilitated when students interact with multiple models that differ in perceptual features causing students to continuously rethink and ultimately generalize the concept* (Dienes, cited in Post & Reys, 1979).

As you will see in the next section, models should be used as a way to understand and generalize mathematical ideas; that is, *models are a means to the mathematics, not the ends* (Post, 1981; Clements, 1999).

Modeling as the Means to Understanding Mathematics, Not the End

Models are mental maps mathematicians use as they solve problems or explore relationships. For example, when mathematicians are thinking about a number, they may have a number line in mind. They think about where the numbers are in relation to one another on this line, and they imagine moving back and forth along the line.

(Fosnot & Dolk, 2002, p. 73)

In this case a mathematician's model is a well-established "mental map." However, as students are developing their understanding of concepts, they will physically construct models to solve problems and represent concepts. Over time, students should move from the need always to construct or use physical models to carrying the mental image of the model, while still being able to make a model as they learn new concepts or encounter a difficult problem. This interview with Jared, a third grader, makes this case in point.

Student Interview—On Using Models

Interviewer: I know that mathematicians use models, but sometimes kids in school are uncomfortable using them.

Jared: I think it's pretty comfortable because sometimes if you try to do it in your head its gets harder and if you use like blocks or diagrams or anything it will help a lot. Sometimes my favorite thing is like a number line or a T-table or something. That's what I do a lot.

Interviewer: It's nice to hear that you are comfortable to draw or get other materials or that kind of thing.

Jared: Yeah, because it helps you do the questions a lot better.

Interviewer: Well, you can see it, right? It's not just words on a page.

Jared: Yeah, because if you do it in your head you can't do it as good. Sometimes I first use blocks. Then I sort of sometimes imagine blocks. So now I sort of do it in my head.

Interviewer: Wow!

Jared: So I can imagine blocks and I can do it without real blocks and I can do it in my head now. Because I did it with blocks and got it in my head I can do it pretty easy now.

Interviewer: Why can you do it in your head now?

Jared: Because I used blocks a lot in first and second grades, and since I did it a lot it sort of got stuck in my head.

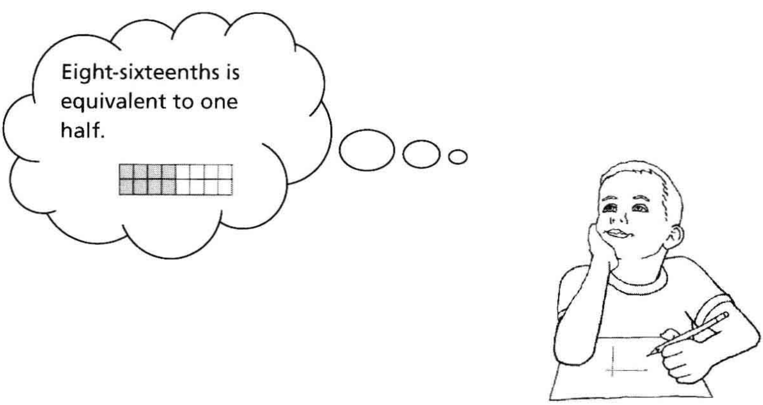
Interviewer: What happens when a problem gets hard?

Jared: When the problems get like harder and harder, when they are really hard I sometimes need to draw or something.

(VMP, student interview, 2007)

One suspects that Jared's confidence in solving problems and using either "pictures in his head" or physical models are the result of Jared having

Figure 1.1 Jared is using a mental image to solve the problem



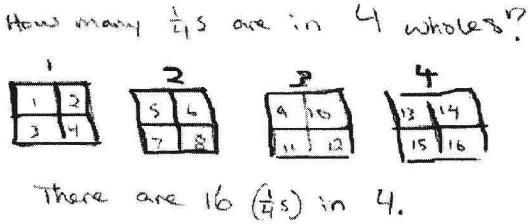
experience with a variety of models over time as he developed his understanding of the concepts.

In addition, one would expect teachers to provide experiences in which students are encouraged to look for patterns and relationships, make and explore conjectures, and to use what they learn from their models to generalize concepts. Let's explore Kelyn's response to a division problem in Figure 1.2 to make this case.

Figure 1.2 Kelyn's model is a good conceptualization of this division problem

$4 \div \frac{1}{4}$ is closest to?

A. 10
B. 1
C. 0
D. 15



The evidence in Kelyn's solution leads one to believe that she understands that there are four one-fourths in a whole and 16 one-fourths (written in Kelyn's response as "16 $\left(\frac{1}{4}\text{s}\right)$") in four wholes. Kelyn's model is a good conceptualization of this division problem. The challenge for instruction is to ask questions or present situations that *capitalize* on this conceptualization to lead to a generalization about dividing a whole number by a proper fraction.

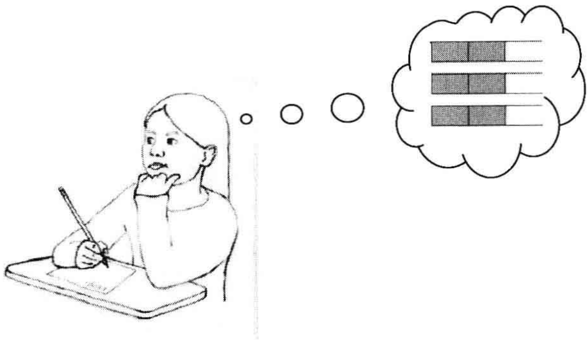
Here are some examples of questions that one might ask Kelyn, to help her move to a more generalized understanding of the division of fractions.

- Your model shows that there are four one-fourths in every whole. How many fourths do you think there are in 5 (or in 6 or in 10 or in 100)?
- How many thirds, fifths, or sixths are in 4 (or in 5 or in 6 or in 100)?
- What patterns do you see?
- Make and test a conjecture about the patterns that you see (giving Kelyn the chance to say "I noticed that . . .").

With additional questioning and exploration by Kelyn, she should be able to extend her understanding of unit fractions to division of whole numbers by proper fractions that are not unit fractions (Figure 1.3).

- How many three fourths are in 3 (or in 6 or in 9 or in 12)?
- How many two thirds are there in 2 (or in 4 or in 6 or in 8)?
- Make and test a conjecture about the patterns that you see.

Figure 1.3 Kelyn's mental image of $3 \div \frac{2}{3}$



Kelyn's Thinking

> Well, I know that there is one two-thirds in each whole with one third left. If there are three wholes, then there are three two-thirds with three one-thirds left. Two of the thirds make another two thirds. Now I have four two-thirds and I am left with one third. One third is half of two-thirds. I think the answer is $4\frac{1}{2}$.

Jared's transition back and forth between mental models and physical models, and the potential for Kelyn's teacher to capitalize on her conceptualization of the division of fractions to make a generalization about division makes an important point. *Models are a means to the mathematics, not the end* (Post, 1981; Clements, 1999). "*The provision of multiple experiences (not the same many times) using a variety of materials, is designed to promote abstraction of the mathematical concept*" (Dienes, cited in Post, 1981).

Using models, regular probing, and asking students to explain their thinking or demonstrate their models, should play a key role in instruction as students are solving problems and building their understanding of part to whole relationships, the relative magnitude of fractions (equivalence, comparing and ordering fractions), or fraction operations.

One important point needs to be made before we proceed to the details about the features of models. *The use of models (both teacher and student generated) should permeate instruction; not just be an incidental experience, but a way of thinking and learning for students.*

- Students should have the opportunity to solve problems in which they interact with models (Figure 1.13: find $\frac{3}{8}$ of the figure);
- Students should have the opportunity to solve problems by generating their own models (Figure 1.2: Kelyn's division solution);
- Students should have the opportunity to use models to develop understanding of concepts (e.g., use a model to show that $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent);

- Teachers should build on student generated models to help them generalize mathematical ideas by asking students to explain their models and respond to probing questions that capitalize on understandings in their models (e.g., Kelyn's division models).

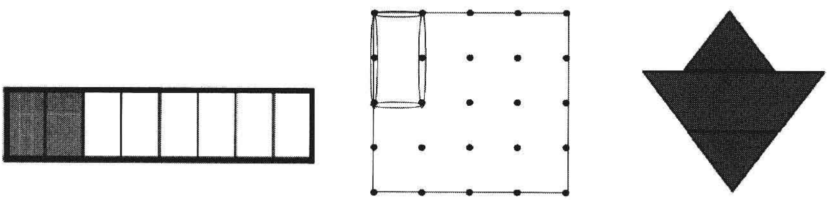
GO TO Activities and questions related to using models to generalize concepts are found in: Chapter 4 (Partitioning)—section focusing on Using Partitioning to Generalize Concepts (p. 75); Chapter 7 (Density of Fractions)—question 2 in Looking Back; Chapter 8 (Equivalent Fractions and Comparisons)—question 2 in Looking Back; Chapter 9 (Addition and Subtraction of Fractions)—questions 2, 4, and 5 in Looking Back; Chapter 10 (Multiplication and Division of Fractions)—questions 2 and 6 in Looking Back.

Features of Area and Set Models, and Number lines

There are three different types of models that students will interact with, use to solve problems, and use to generalize concepts related to fractions—area models (regions); set models (sets of objects); and number lines. Samples of each type of model are found in Figures 1.4 to 1.6.

Using area models involves thinking about part to whole relationships. Area models that students typically interact with in mathematics programs and instruction include objects or drawings such as grids, geoboards, paper folding, and pattern blocks. Some examples of area models are pictured in Figure 1.4.

Figure 1.4 Area model samples



Using set models involves thinking about a fractional part of a set of objects. Set models that students typically interact with in mathematics programs and instruction include collections of common objects found in a classroom (e.g., buttons, candies, and marbles). Some examples of sets of objects are pictured in Figure 1.5.

Using a number line involves thinking about the distance traveled on a line or the location of a point on number lines, rulers, or other measurement tools. Some examples of number lines are pictured in Figure 1.6.

Figure 1.5 Set model samples—sets of objects arranged in an array (a set of apples in a 6×5 array), scattered (a set of 6 marbles), and in a composite set (24 eggs in 2 sets of 12 eggs)

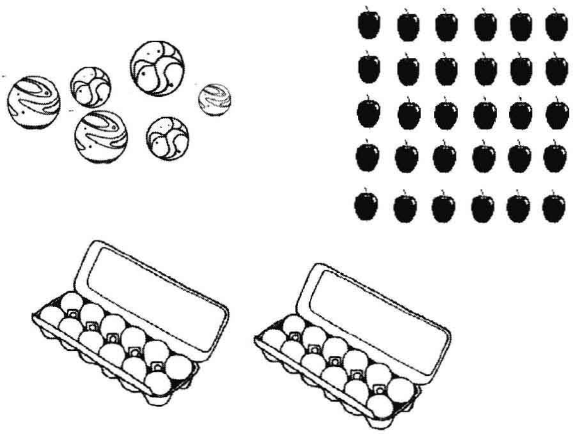
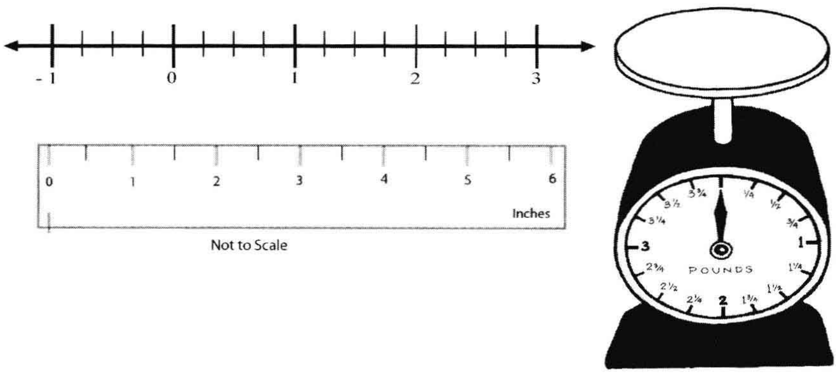


Figure 1.6 Number line from −1 to 3, a ruler, and a scale



Why Models Differ in Challenges

According to research, area and set models, and number lines differ in the challenges that they present students (Hunting, cited in Bezuk & Bieck, 1993; VMP OGAP, personal communication, 2005, 2006, 2007).

Why the models differ in challenge, and why it is important for students to encounter the three types of models are related, in part, to three aspects of the models when working with fractions: