THE ALGEBRA OF ELECTRONICS

by

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PREFACE

In recent years, there has been a renaissance of our forefathers' do-it-yourself philosophy. Fortunately, this trend is found in the intellectual realm, as well as in craftwork. Thousands of persons have learned algebra and calculus by home study of suitable texts.

This book is addressed to these people, with their healthy intellectual curiosity. It is addressed equally to electronic technicians, and TV and radio servicemen, who have a practical knowledge of circuits and wish to acquire understanding.

The book starts with direct current, to introduce the basic concepts without confusing detail. Networks of resistance are discussed topologically, in terms of trees, branches, links, and loops. Mesh and nodal analysis are presented as special cases, for which the network equations can be written by inspection, in a form that continues to be valid for the general AC case. This leads into the study of determinants and the solution of simultaneous equations. Practical solution methods are emphasized.

The fourth chapter treats of general properties of networks, and their representation as T-networks, Π-networks, and "black boxes."

The transition to alternating current problems is made via chapters on capacitance and inductance, developed from fundamentals. Simple tuned circuits follow, and lead into the concept of impedance, and its various representations in terms of phase angles and complex numbers. The arithmetic and algebra of complex numbers is treated in detail.

A major chapter on general AC networks elaborates the treatment of mutual inductance, and clarifies the question of the algebraic sign of mutual inductance in multi-coil assemblies. The behavior of air-core and iron-core transformers is thoroughly explained, and various equivalents are analyzed.

Chapter X is devoted to the analysis of specific circuits, such as double-tuned interstage, FM discriminator, bridged-T, twin-T, and an RC ladder used in phase shift oscillators.

Chapter XI discusses impedance matching sections and the various phenomena associated with matching, mismatching, filtering, and insertion loss.

The study of diodes as nonlinear elements leads to triodes, and their linearized approximation in terms of amplification factor and mutual conductance. Amplifiers are treated as specific circuits involving thermionic tubes and transistors, and finally as general active "black boxes." Because the practical limits to amplification depend upon noise, a chapter is devoted to this subject.

The text concludes with a study of modulation, demodulation, and distortion, explained in terms of frequency components and Fourier series. Problems, hints, and answers, close the book.

The basic topological concepts of electric networks in the early chapters follow the philosophy of Professor Ernst Guillemin, who has written several excellent books for a more advanced audience. All of us who are interested in either network research or teaching owe a debt of gratitude to Professor Guillemin for his unceasing output of ideas and enthusiasm.

C.H.P.

Silver Spring, Maryland
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Chapter I

DIRECT CURRENT

1-1 Voltage. It is a familiar fact of electrostatics that unlike charges (positive and negative) attract each other. A force must be applied to move them apart; work must be done against the force that tends to pull the charges together. This work becomes potential energy associated with the separated charges, just as the work done in lifting a weight becomes potential energy.

Consider two conducting bodies, say metal plates, not quite in contact. Let one plate carry the charge Q and the other, $-Q$. (These charges can be obtained by various means. One way is to rub a glass rod with silk, and then "wipe" the charge off the rod onto the metal plate.) We now separate the plates, applying the necessary force by way of insulating handles. The work we do becomes electrical potential energy, and we say that the "potential difference" between the plates has increased. Conversely, if we have charged separated plates, we can let their electrical attraction pull them together, and do work for us, such as lifting a weight. The total amount of work they can do, by going completely together, is the potential energy of the system. For a given separation, this energy is greater, the greater the charge on the plates. The potential energy per unit of charge (W/Q) is called the "potential difference," or *p.d.*, between the plates, and is measured in *volts*. The charge Q is measured in *coulombs*, and the energy in *joules*. (The practical unit of charge, the coulomb, is approximately the charge on 6×10^{18} electrons. The joule is more familiar as the *watt-second*; 1 kilowatt-hour equals 3.6 million joules.)

The potential energy of our pair of separated plates can be reduced by letting them get closer together; it will also be reduced if the insulating handles are imperfect, and some of the charge "leaks" from one plate to the other, urged to do so by the attractive force between the unlike charges. In this case, the used-up potential energy shows up as heat (thermal energy). This will be treated in detail a little later. In either case, loss of potential energy means a lowering of the p.d. between the plates. Now if the plates are connected to the terminals of a battery or a generator, the p.d. will be held constant, even though we have used some energy. The used energy was, of course, supplied by the battery. But to keep the p.d. constant, additional separated charge must have been supplied to the plates. This is obvious when the energy loss was due to charge loss; the case of energy loss due to motion of the plates will be discussed in a later chapter (V). Thus the battery or generator has some sort of internal "force" that tends to push positive charges out one terminal, and negative out the other, to supply positive and negative charges to the plates. Such an electrical separating force is called an *electromotive force* (emf) and is measured by the potential difference it maintains between the terminals, hence it is measured in volts. We see, then, that the term "voltage" is used for both electromotive force (a cause), and potential difference (a result) even though these quantities are logically different. In fact, if we short-circuit a dry cell by connecting its terminals together with a good conductor, we do not affect the emf of the cell, but we can no longer have a p.d. between the terminals. Indeed, after a short while, we will no longer have a cell!

In problems involving electric currents in equilibrium with their driving forces, as we are throughout this book, it is best to think of an emf as a *source* voltage, and a p.d. as a *resulting* voltage across any device which is not a source.

1-2 Current. Electrostatics is the study of electricity when the charges are essentially at rest. Most practical usages of electricity involve the flow of charges through a conductor, analogous to the flow of water through a pipe. In the water analog, the flow is measured by the quantity (gallons) passing a given point in a unit of time (minute). In a river, an open pipe supplied by nature, this flow of water is called a *current*, and is measured in gallons per minute, or millions of gallons per hour, or some other convenient combination of quantity and time. By analogy, the flow of electric charge is called electric *current*, and is measured in *coulombs per second.* For convenience, this unit of current has been given a name of its own: *ampere.* Thus a current of ten amperes means the flow of ten coulombs of charge each second. (Note that "current" is the "flow of charge." It is logically redundant to say "a current flows through a wire.")

When the flow of charge is steady, the current is $I = Q/t$. When the flow is not steady, the instantaneous current is the instantaneous rate of charge flow and is given by the time derivative: $I = dQ/dt$.

Recall the discussion on potential energy (W) and potential difference (V). Since potential difference is the energy per unit charge, $V = W/Q$, we can write $W = VQ$ for the work done by the charge Q in "falling" through a potential difference V. When the p.d. is constant, we can differentiate with respect to time and find

$$P = dW/dt = V\,dQ/dt = VI$$

since *power* is the rate of doing work. (Power is measured in *watts*, or joules per second.) This equation, $P = VI$, is one of the basic relations in the study of electricity.

Another basic relation was discovered by G. S. Ohm in 1827. Ohm found experimentally that if the voltage across a wire (the engineer's way of saying "the potential difference between the two ends of a wire") was increased, the current through the wire increased proportionately. That is, the ratio V/I is constant, for a given piece of wire. This ratio was given the name *resistance* ($R = V/I$) and is measured in *volts per ampere*. The unit of resistance, one volt per ampere, has been named the *ohm* to honor this pioneer electrician.

Ohm also found that if a second wire, identical with the first, was connected to offer the current an additional path, the current was doubled. That is, the two wires connected side-by-side, or in parallel, each passed as much current as the first wire by itself. This implies that the currents in the alternative paths can be computed independently, and added to find the total current.

He also found that if the two wires were connected in *series* (end-to-end) so that they carried the same current, the necessary voltage was doubled. That is, the voltage across an end-to-end set of wires is the sum of the individual voltages. In hindsight, these findings seem obvious, for a four-foot length of wire is the same thing whether we consider it as one four-foot length, two two-foot lengths, or four one-foot lengths, etc. Similarly, a fat wire can be conceived as a bundle of thin wires side-by-side.

Extension of the above experiments and logic showed that the resistance of a conductor is proportional to its length, and inversely proportional to its cross-sectional area, $R \propto l/A$. The constant of proportionality (ρ) that makes this relation an equation, $R = \rho l/A$, is a characteristic of the material of which the conductor is made, and is called the *resistivity*. The resistivity of a material varies with temperature, but is independent of the shape and size of the conductor. Since $\rho = RA/l$, its unit is (ohms) times (square centimeters) divided by (centimeters), which simplifies to (ohms) times (centimeters), or ohm-cm. For good conductors, such as the metals, the resistivity is only a few millionths of an ohm-cm, so is commonly listed in handbooks in microhm-cm.

The combination of the power equation ($P = VI$) and Ohm's law ($V = RI$) yields by algebraic substitution, $P = I^2R = V^2/R$ as alternate ways of computing power.

Example.

An electric lamp that draws 100 watts at 120 volts has a resistance $R = V^2/P = 144$ ohms, and draws a current of $I = V/R = 0.833$ ampere, or $I = P/V = 0.833$ ampere.

1-3 Resistors in Combination. A brief digression on "things" and "representation" of things is in order at this point. We have seen that resistance is an abstraction; it is the ratio of a voltage to a current. A "device" which is used because it has resistance, is called a *resistor*. The schematic symbol —/\/\/\/— used in *wiring* diagrams represents a resistor, and connecting lines represent actual wires. In this case the diagram is a conventional "picture" of an actual assembly of concrete "things," such as resistors and batteries. On the other hand *circuit* diagrams also represent abstractions, such as combinations of voltage and resistance. Real resistors can overheat, or burn up, or have peculiar unexpected properties,

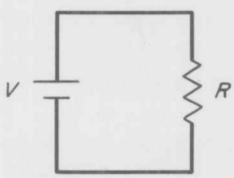

FIG. 1.1.

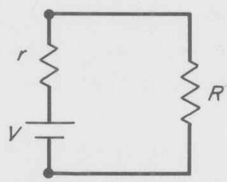

FIG. 1.2.

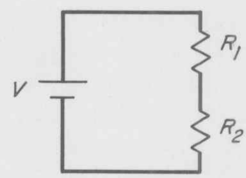

FIG. 1.3.

whereas the abstract resistance of a theoretical diagram is a well-behaved mathematical quantity. The circuit diagrams, or networks, in textbooks represent abstract concepts of resistance, voltage, etc. Circuit theory is an intriguing mathematical game, whose answers are always perfect. If the idealized "mathematical circuit" turns out to be a reasonable representation of the properties of a real device, then the answers of the game will also be a reasonable approximation to what the real device will do.

The mathematics of circuit theory is perfect and exact; the engineer's big problem is to make sure that his mathematical model truly represents the device he is building. Stray wiring capacitance, lead inductance, and leaky insulation do not show on a *wiring diagram*, but must be included in an *abstract circuit diagram* if the engineer wants his analysis to give him good results. For example, Fig. 1.1 as a wiring diagram represents a resistor connected across a battery. Every practical man knows that a battery cannot deliver an infinite current; if the resistance is made too small, the voltage across it will not be V, but will be less. For our purposes, however, Fig. 1.1 is an abstract diagram, and the voltage across the resistance is V no matter how much current is drawn. For "reasonable" currents, the two interpretations of the figure are equivalent. For "large" currents, we shall see later that the voltage symbol —||— by itself is not sufficient to represent a real battery. In fact, a real battery has *internal resistance*, it behaves like a series combination of a perfect battery and a resistor, as in Fig. 1.2. This representation would never be used in a *wiring diagram*, for some technician would be sure to follow it literally and install a resistor r!

Ohm's law tells what happens when a single resistor is connected across a voltage source, as in Fig. 1.1: $I = V/R$. How do we find the current when two known resistors appear in series across a voltage, as in Fig. 1.3? If R_1 and R_2 were pipes carrying water, we would not hesitate to say that the same water flows through both pipes; i.e., they carry the same current. In the electrical case, this conclusion is still true. The argument by analogy does not *prove* the electrical case, it merely *suggests* it. The proof, however, follows the same lines for both water and electric charge. We assume that water cannot suddenly appear or disappear; it must all be accounted for. If the flow through R_1 is different from that through R_2, it can be due only to a leak at the connection, hence a third "pipe" should appear in the diagram. Similarly, electric charge is conserved, and cannot appear, disappear, or pile up at a connection. A current can split at a "fork in the road," but where there is only one path, as in Fig. 1.3, the current must be the same at all points in the circuit.

Now by Ohm's law, the voltage across R_1 is

$$V_1 = IR_1$$

and that across R_2 is

$$V_2 = IR_2$$

where the same symbol I appears in each equation, because it represents the same current in both cases. The total voltage, or p.d., between the upper terminal of R_1 and the lower terminal of R_2 is the sum

$$V = V_1 + V_2 = I(R_1 + R_2)$$

as is suggested by our old friend, the water flow analog, with p.d. analogous to water pressure. Since $V = IR$, it is apparent that the net resistance of the series combination is

$$R = R_1 + R_2$$

This result can be deduced rigorously by appealing to the conservation of energy. The power dissipated in the resistances is

$$P_1 = I^2R_1, \quad P_2 = I^2R_2$$

and the total power supplied by the voltage source is therefore

$$P = P_1 + P_2 = I^2(R_1 + R_2) = I^2R$$

The argument applies to any number of resistances in series:

$$R = R_1 + R_2 + R_3 \cdots$$

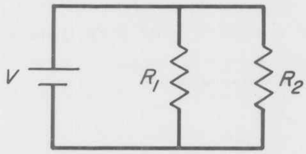

FIG. 1.4.

Similarly, if we connect two resistances in parallel, i.e., *across the same voltage*, as in Fig. 1.4, the respective currents are

$$I_1 = V/R_1 \quad \text{and} \quad I_2 = V/R_2$$

Our fundamental hypothesis on the conservation of charge requires the total current to be

$$I = I_1 + I_2 = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V/R$$

so that the total current is the same as would be drawn by a single resistance R computed from

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \text{or} \quad R = \frac{R_1R_2}{R_1 + R_2}$$

Example.

Two resistances of 50 ohms and 100 ohms yield a *series* resistance of 150 ohms; and a *parallel* resistance of $33\frac{1}{3}$ ohms.

Again, our formula can be extended to any number of resistances in parallel:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

For more than two resistances in parallel, the simplest computation is to use the equation as shown: add the reciprocals of the resistances, and take the reciprocal of the sum, i.e.,

$$R = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots}$$

The formula corresponding to $R_1R_2/(R_1 + R_2)$ is not convenient for more than two resistances.

Example.

The parallel combination of 6, 4, 3, and 2 ohms has the resistance R given by

$$\frac{1}{R} = \frac{1}{6} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{2}{12} + \frac{3}{12} + \frac{4}{12} + \frac{6}{12} = \frac{15}{12}$$

so that $R = 12/15 = 0.8$ ohm.

A *series-parallel* combination, such as shown in Fig. 1.5, requires a piecemeal analysis using both formulas. The series combination of 10 and 20

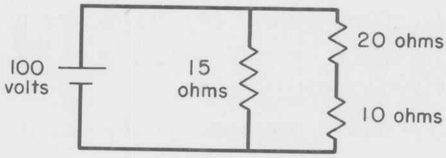

FIG. 1.5.

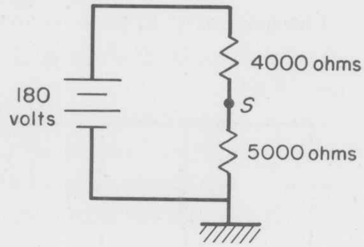

FIG. 1.6.

is 30 ohms; this 30 ohms in parallel with 15 ohms yields a net resistance of $30 \times 15/45 = 10$ ohms. The battery current is therefore 10 amperes, of which $100/15 = 6\frac{2}{3}$ is in the 15-ohm branch, and the remaining $3\frac{1}{3}$ is in the 30-ohm branch. This $3\frac{1}{3}$ amperes produces a *voltage drop* (p.d.) of $66\frac{2}{3}$ volts across the 20-ohm resistance, and $33\frac{1}{3}$ volts across the 10 ohms. Note that the 10–20 series combination divides the supply voltage in that ratio.

This voltage-dividing property of resistors in series is often used in radio receivers, where there is a supply of, say, 180 volts for the plate of a tube and, say, 100 volts is wanted for the screen grid. If we connect a voltage divider as in Fig. 1.6, we will have 100 volts across the lower resistor, or between point S and ground. The 100 volts at S is, however, the *no-load*, or *open-circuit voltage* (OCV). If the screen draws, say, 4 ma (0.004 ampere) at 100 volts, the extra 4-ma current through 4000 ohms would produce an additional voltage drop of 16 volts. Instead of 100 volts at S, we would have only 84 volts; but at 84 volts the screen would draw less than 4 ma. If we know the screen current at 84 volts, we can recompute the actual voltage at S; repeating this procedure would give a set of successive approximations that would finally steady down to the correct answer—but what a lot of work! In any case, we don't really want to know what voltage will appear at S; we want to know what resistances to use in the voltage divider so that we will have 100 volts at S, with a current drain of 4 ma.

Since 4 ma at 100 volts represents a load of 25,000 ohms, our complete circuit is as in Fig. 1.7. If we wish to keep $R_2 = 5000$, the total current is $100/5000 + 0.004$, or 24 ma. This current must produce an 80-volt drop across R_1, or $R_1 = 80/0.024 = 3333\frac{1}{3}$ ohms.

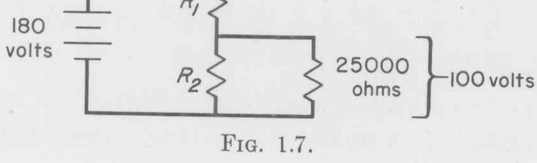

FIG. 1.7.

Problem.

Leave R_1 at 4000 ohms, and compute the approximate value of R_2. (Answer: 6250)

Let us now investigate the general behavior of a voltage divider (Fig. 1.8) by using algebra (instead of arithmetic, which is used for particular situations). Let the load resistance be R_L, the voltage across the load, V_L, and the current through the load, I_L. We are interested in the way the

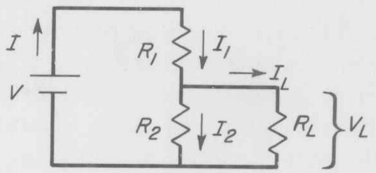

FIG. 1.8.

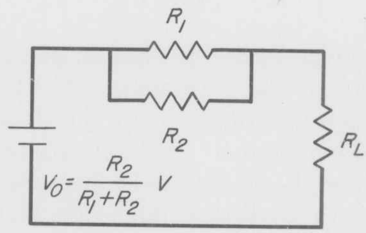

FIG. 1.9.

output voltage varies with load. Since all current must be accounted for, we have $I = I_1 = I_2 + I_L$. We have also

$$V_L = I_L R_L = I_2 R_2$$

$$V = V_L + I_1 R_1$$

These last two equations give

$$I_1 = (V - V_L)/R_1 \quad \text{and} \quad I_2 = V_L/R_2$$

which, substituted into the first equation, yields

$$\frac{V}{R_1} - I_L = V_L\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_L\,\frac{R_1 + R_2}{R_1 R_2}$$

and, finally,

$$V_L = \frac{R_2}{R_1 + R_2}\,V - \frac{R_1 R_2}{R_1 + R_2}\,I_L$$

But this last equation also describes the behavior of the circuit of Fig. 1.9, and furthermore, the voltage V_o of Fig. 1.9 is the OCV of the original voltage divider! The implication of this result is, that if we have two boxes containing the alternate arrangements of Fig. 1.10, there is *no external experiment that can distinguish one box from the other.*

The OCV is obviously the same for the two boxes. The short-circuit currents are readily computed to be the same, also.

$$I_S = V/R_1 = V_o/R_0$$

For the purposes of *circuit theory*, then, the two boxes are *externally* equivalent; as *wiring diagrams* they would differ, for one battery would run down without a load. Even with a perfect battery, the boxes are not *internally* equivalent, for the total power dissipation differs. But to all *external* appearances, the boxes behave identically.

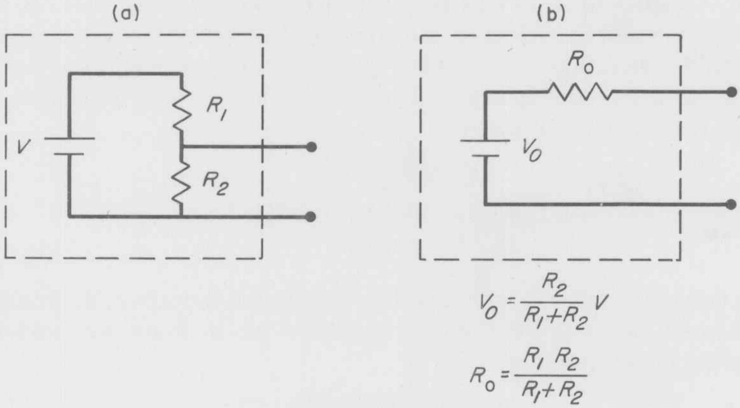

FIG. 1.10.

This example is a particular illustration of Thevenin's theorem. The general case will be discussed in later chapters.

The power delivered to a load by a voltage divider exhibits an interesting property. The load current is

$$I_L = V_o/(R_0 + R_L)$$

so the load power is

$$P = I_L{}^2 R_L = V_o{}^2 R_L/(R_0 + R_L)^2$$

The load power goes to zero as R_L goes to zero, by virtue of the R_L in the numerator. As R_L increases without limit, the power again goes to zero by virtue of the square in the denominator. The value of R_L for maximum power can be found by differentiation. Now

$$\frac{dP}{dR_L} = V_o{}^2 \frac{(R_0 + R_L) - 2R_L}{(R_0 + R_L)^2}$$

which vanishes, indicating the maximum of P, when $R_L = R_0$, and giving

$$P_{\max} = V_o{}^2/4R_0$$

This maximum power is called the "available power." Note that it is obtained by matching resistances, i.e., by making the load resistance equal to the *internal resistance* of the source.

All real sources (batteries, generators, amplifiers, etc.) are imperfect—they have internal resistance and are equivalent to the "black box" of Fig. 1.10b. In the ideal case of linear behavior, the internal resistance is constant (independent of current) and can be found experimentally as the ratio: (open-circuit voltage) ÷ (short-circuit current).

1-4 Graphical Description of Source and Load. Let us reconsider our previous problem of a load R_2 connected across the source Fig. 1.10b. We shall take the output voltage (V) and current (I) as variables for description of the source and load behavior. For any current I the output, or terminal voltage, is

$$V = V_o - R_0 I \tag{1-1}$$

whereas the load resistance specifies the relation

$$V = R_L I \tag{1-2}$$

These equations must *both* be satisfied, and by the same pair of values V, I. The solution of these simultaneous equations can be found by substituting one in the other, yielding

$$I = V_o/(R_0 + R_L) \tag{1-3}$$

$$V = V_o R_L/(R_0 + R_L) \tag{1-4}$$

This problem can also be solved graphically. Equation (1-1) says that the allowable pairs of V, I are represented by the points of the straight line of Fig. 1.11. The line is constructed by connecting its extreme points: (a) open-circuit voltage and zero current and (b) zero voltage and short-circuit current. The slope of the line is $-R_0$. Since Fig. 1.11 shows the

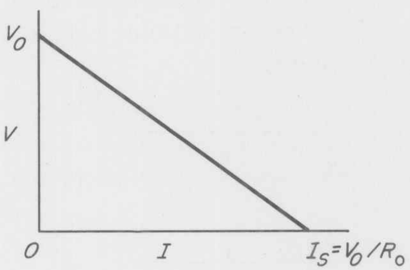

FIG. 1.11.

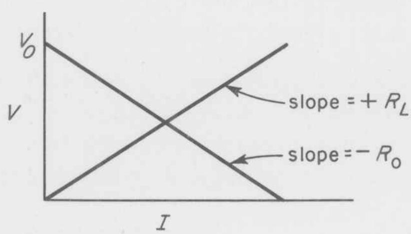

FIG. 1.12.

terminal voltage for a specified current, or vice versa, it is a graphical description of the electrical properties of the source. The load can similarly be described by plotting Eq. (1-2); superposing the load and source lines on the same graph gives Fig. 1.12; the intersection satisfies both equations and is the point specified by Eqs. (1-3) and (1-4).