Problems in Theoretical Physics

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Preface

This text incorporates problems which have been used for several years at seminars in courses in classical mechanics, electrodynamics, quantum mechanics, and statistical physics and thermodynamics at the T. G. Shevchenko State University in Kiev.

The text draws largely on the *Course of Theoretical Physics* by L. D. Landau and E. M. Lifshitz, but also makes use of other textbooks and handbooks recommended for the university course in theoretical physics. Some of the problems have been taken from published problem books listed at the end of this book, but many are original.

The student will be able to solve the problems if he has a good knowledge of the fundamentals of theoretical physics, which are briefly outlined in each section of this book. All the problems use the International System of Units (SI).

The section on classical mechanics was compiled by A. M. Fedorchenko, on electrodynamics by V. I. Sugakov, on quantum mechanics by O. F. Tomasevich, and on statistical physics and thermodynamics by L. G. Grechko.

The Authors

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SECTION I

Classical Mechanics

The mechanics of systems with a finite number of degrees of freedom. In mechanics a particle is a material body of mass m whose position in space is determined by three coordinates.

The mechanical state of a system of n particles is characterized by $3n$ coordinates and the $3n$ time derivatives of these coordinates. The law involving changes in the state of a mechanical system in time is defined by Newton's equations

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \quad i = 1, 2, \ldots, n \tag{I-1}$$

where $\mathbf{F}_i$ is the resultant of all the forces acting on the ith particle; these include both internal forces (those originating in the particles of the system) and external forces (those having a source outside the system and such as considered given at any instant of time).

From the standpoint of mathematics equation (I-1) is a system of $3n$ differential equations. For this reason the basic problem of mechanics consists in finding a solution for this system. We know from the theory of differential equations that to find an unambiguous solution of the system we must indicate $6n$ values $\mathbf{r}_i^0$, $\dot{\mathbf{r}}_i^0$ at a definite instant of time. In short, the mechanical state of a system at any subsequent time is determined by its initial mechanical state $\mathbf{r}_i^0$, $\dot{\mathbf{r}}_i^0$ and by the forces acting on each particle in the system.

Equations (I-1) are valid only in inertial frames of reference. An inertial frame of reference is one in which a particle free from forces, i.e. an isolated particle, is in uniform rectilinear motion. The first law of mechanics states that such frames do exist.

Forces acting between two particles are represented by the formula

$$\mathbf{F}_{ij} = F(r_{ij}) \frac{\mathbf{r}_{ij}}{r_{ij}}$$

which reflects the following properties (Fig. 1):

(1) $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$;

(2) $\mathbf{F}_{ij} \parallel \mathbf{r}_{ij}$;

(3) the magnitude of the force depends only on the distance between the two particles.

Classical mechanics rests on the three laws of Newton, which were deduced from experiments and observations of mechanical motion. All other assertions and laws of mechanics, valid for specific conditions and specific models, are corollaries of these three laws.

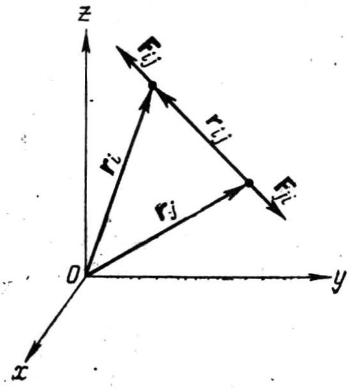

Fig. 1

In a noninertial frame of reference (one moving with acceleration) equations (I-1) do not hold. But we can preserve the form of equations (I-1) by introducing what are called forces of inertia, whose origin cannot be explained by the action of any specific particles. The forces are due to the fact that the frame of reference moves with acceleration. The equation of motion for a particle in a noninertial frame of reference is

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{iner}}$$

where $\mathbf{F}_{\text{iner}} = -m(\ddot{\mathbf{R}}_0 + [\dot{\boldsymbol{\omega}} \times \mathbf{r}] + [\boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{r}]] + 2[\boldsymbol{\omega} \times \dot{\mathbf{r}}])$ is the force of inertia. $\ddot{\mathbf{R}}_0$ is the acceleration of the coordinate origin and $\boldsymbol{\omega}$ is the angular velocity of this frame [see formulas (I-23) and (I-24)].

If we proceed from the second law of Newton (I-1) and the first property of the forces of interaction (see above), we can prove that the time derivative of the momentum vector of a system of particles equals the sum of all the external forces, $\mathbf{F}_{ext}$:

$$\frac{d\mathbf{p}}{dt}=\mathbf{F}_{ext} \tag{I-2}$$

where $\mathbf{p}=\sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i$; n is the number of particles in the system.

If the system is closed, i.e. $\mathbf{F}_{ext}$ equals zero, equation (I-2) gives us the law of conservation of momentum:

$$\mathbf{p}=\text{constant}$$

If we introduce the notion of the centre of mass of a system

$$\mathbf{R}=\frac{\sum_{i=1}^{n} m_i\mathbf{r}_i}{M}$$

where $M=\sum_{i=1}^{n} m_i$, equation (I-2) takes the form

$$M\ddot{\mathbf{R}}=\mathbf{F}_{ext} \tag{I-3}$$

If the system is closed, it follows from equation (I-3) that

$$\dot{\mathbf{R}}=\text{constant}$$

Thus, the velocity of the centre of mass of a closed system remains constant.

From equation (I-2) we can deduce the law of motion of a body having variable mass, i.e. the law of jet propulsion. In the simplest case, if the main body (of mass m) is losing or gaining mass, the law of jet propulsion (Meshcherskii's formula) takes the following form:

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}_{ext}+\frac{dm_1}{dt}\mathbf{u}_1-\frac{dm_2}{dt}\mathbf{u}_2 \tag{I-4}$$

where m_1 is the mass gained, $\mathbf{u}_1$ is its velocity relative to the main body, and m_2 and $\mathbf{u}_2$ are the respective values for the lost mass.

Proceeding from the second law of Newton (I-1) and the first two properties of the forces of interaction, we can prove that the time derivative of the angular momentum of a system of particles equals the sum of the moments of all the external forces, **N**:

$$\frac{d\mathbf{L}}{dt} \equiv \dot{\mathbf{L}} = \sum_{i=1}^{n} [\mathbf{r}_i \times \mathbf{F}_i] = \mathbf{N} \qquad \text{(I-5)}$$

where $\mathbf{L} = \sum_{i=1}^{n} m_i [\mathbf{r}_i \times \dot{\mathbf{r}}_i]$.

We must bear in mind that the radius vectors $\mathbf{r}_i$ of the particles in the system, which vectors enter into the definitions of the angular momentum and the moment of an external force, must issue from the same point because both depend on the choice of the coordinate origin.

Newton's third law makes it possible to introduce the concept of the potential of a force according to the formula

$$\mathbf{F}_{ij} = -\text{grad}_i\, U(r_{ij}) \qquad \text{(I-6)}$$

where the potential $U(r_{ij})$ depends only on the distance between the interacting particles.

We can use the potential concept to prove the following theorem on the basis of Newton's laws of motion: a change in the mechanical energy of a system equals the work done by external forces, i.e.

$$d\left(K + \frac{1}{2}\sum_{ij} U_{ij}\right) = dA \qquad \text{(I-7)}$$

where by definition $K = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^2$ and $dA = \sum_{i=1}^{n} (\mathbf{F}_{\text{ext}})_i\, d\mathbf{r}_i$.

The law of conservation of energy holds for closed systems:

$$E = K + \frac{1}{2}\sum_{ij} U_{ij} = \text{constant} \qquad \text{(I-8)}$$

If a part of the external forces has a potential V, we can write formula (I-7) as

$$\frac{d}{dt}\left(K + \frac{1}{2}\sum_{ij} U_{ij} + V\right) = -\frac{\partial V}{\partial t} + \sum_{i=1}^{n} (\dot{\mathbf{r}}_i \cdot \mathbf{f}_i)$$

where $\mathbf{f}_i$ is a nonpotential force.

Thus, a closed mechanical system always has seven integrals of motion (seven functions of coordinates and velocities), which remain constant upon motion. In the general case the number of integrals of motion, which do not depend on time, is $2k - 1$ for a closed system, where k is the number of degrees of freedom. The seven aforementioned integrals of motion play a special role in physics. There are two main reasons for this. First, these integrals of motion always exist regardless of the number of particles in the system (for a single particle not all are independent). Second, their existence can also be proved by the fundamental properties of space-time. For instance, the law of conservation of momentum follows from the homogeneity of space (all points in space have the same status); the law of conservation of angular momentum follows from the isotropy of space (all directions in space have the same status); the law of conservation of energy follows from the homogeneity of time (all instants of time are equivalent).

The laws of motion have other forms than Newton's. Using the Lagrangian function (or, simply, the Lagrangian) and the generalized coordinates, we can write equations (I-1) in the following form:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q'_i \qquad \text{(I-8')}$$

where $\mathcal{L}$ is the Lagrangian defined as $\mathcal{L} = K - V$ (K is the kinetic energy and V the potential energy of the system); q_i are the generalized coordinates, i.e. any coordinates that satisfy the sole requirement that the Cartesian coordinates (used in the system of Newtonian equations) are at any instant of time uniquely expressed in terms of all the q's:

$$\mathbf{r}_s = \mathbf{r}_s(q_1, \ldots, q_k, t)$$

$$Q'_i = \sum_{s=1}^{n} \left(\mathbf{f}_s \cdot \frac{\partial \mathbf{r}_s}{\partial q_i}\right)$$

where $\mathbf{f}_s$ is a nonpotential force; the subscript k is the number of degrees of freedom.

If there are nonpotential forces in the system but the generalized force corresponding to them can be written as

$$Q_i' = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)$$

where U is a function of the coordinates and velocities, the Lagrange equations of the second kind take the form

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \qquad \text{(I-9)}$$

where $\mathscr{L} = K - V + U$. For example, the Lorentz force

$$\mathbf{f} = e\mathbf{E} + e\,[\dot{\mathbf{r}} \times \mathbf{B}]$$

defined by the equations

$$\mathbf{E} = -\operatorname{grad} \varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A}$$

is a nonpotential force. It can be written as

$$f_x = eE_x + e\,(\dot{y}B_z - \dot{z}B_y) = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)$$

where

$$U = e\varphi + e\,(\dot{\mathbf{r}} \cdot \mathbf{A})$$

The Lagrange equations can be obtained from the variational principle, which states that if we introduce the functional S, called action, according to the formula

$$S = \int_{t_1}^{t_2} \mathscr{L}(t,\ q_i,\ \dot{q}_i)\,dt, \qquad \text{(I-10)}$$

the actual motion will be described by such functions $q_i(t)$ as ensure a minimum of the functional S provided that $q_i(t_1)$ and $q_i(t_2)$ are given.

The Lagrange equations are a system of k second-order differential equations. We know from mathematics that a system of k second-order differential equations can be reduced to a system of $2k$ first-order differential equations. In mechanics this is done by introducing the Hamiltonian function (or, simply, the Hamiltonian), which is a function of the generalized coordinates and momenta. The generalized momenta are defined by the formula

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{I-11}$$

Since the Lagrangian is a quadratic function of the generalized velocities, formulas (I-11) give a (linear and single-valued) relationship between the generalized velocities and the generalized momenta.

The Hamiltonian is related to the Lagrangian in the following way:

$$\mathcal{H}(p_i, q_i, t) = \sum_{i=1}^{f} p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \tag{I-12}$$

All the generalized velocities in the right-hand side of (I-12) must be expressed in terms of the generalized momenta according to (I-11).

The canonical equations of Hamilton are

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{I-13}$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \tag{I-14}$$

Equations (I-13) and (I-14) are a system of $2k$ first-order differential equations.

In some cases the interaction of bodies is of a peculiar nature, the nature of a constraint. Constraints impose certain restrictions on changes in position or velocity. There is a fairly large class of so-called holonomic constraints, i.e. restrictions on position that can be expressed by algebraic equations:

$$f_\alpha(x_1, \ldots, x_n, t) = 0, \quad \alpha = 1, 2, \ldots, s \tag{I-15}$$

These are the equations of constraints.

To solve problems involving constraints we can use the Lagrange equations of the second kind, if we introduce such generalized coordinates as satisfy the equations of constraints automatically, or we can use the Lagrange equations of the first kind in the following form:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{\alpha=1}^{s} \lambda_\alpha \,\mathrm{grad}_i\, f_\alpha \tag{I-16}$$

which must be solved together with (I-15).

If we define the product $-m_i \ddot{\mathbf{r}}_i$ as the force of inertia, we can formulate the d'Alembert principle: a system moves in such a way that on any virtual displacement the work of all the forces, including forces of inertia, at any instant of time equals zero, i.e.

$$\sum_{i=1}^{n} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i)\, \delta \mathbf{r}_i = 0 \tag{I-17}$$

In the absence of constraints this principle gives us the Newtonian equations (I-1). In the case of ideal constraints we get the Lagrange equations of the first kind.

If a system of points (particles) rests while the constraints act on it, the principle (I-17) takes the following form:

$$\sum_{i=1}^{n} (\mathbf{F}_i \cdot \delta \mathbf{r}_i) = 0$$

This equation expresses the principle of virtual displacements, which is the basis of statics. If we add to it the equations of constraints, we can find the condition for the equilibrium of a system of particles.

Solution of equations (I-1) gives us all the information about the mechanical state of a system consisting of any number of particles having an arbitrary law of interaction. However, even the three-body problem (for instance, the problem of three particles interacting via the Coulomb force) poses great mathematical difficulties. For this reason a variety of approximate methods or models that to one degree or another reflect the properties of actual system are used to solve such problems. One is the model of a rigid body. In mechanics a rigid body is a system of particles whose distances from each other remain constant in time. Such a body acts as a single whole while it is in motion.

A rigid body has six degrees of freedom, which can be chosen in the following way. Let us specify an arbitrary