Quantum Field Theory

A Selection of Papers in Memoriam Kurt Symanzik

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Quantum Theory and Pictures of Reality

Foundations, Interpretations, and New Aspects

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Publisher's Preface

Kurt Symanzik was certainly one of the most outstanding theoretical physicists of our time. For thirty years, until his untimely death in 1983, he helped to shape the present form of quantum field theory and its application to elementary particle physics. In memoriam of Kurt Symanzik leading scientists present their most recent results, giving, at the same time, an overview of the state of the art.

This collection was originally published in Vol. 97, 1/2 (1985) of Communications in Mathematical Physics. They range over various interrelated topics of interest to Kurt Symanzik. We hope that making this collection available in an accessible and inexpensive way will benefit the physics community.

The Publisher

Kurt Symanzik

Kurt Symanzik was born November 23, 1923 in Lyck, East Prussia. He grew up in Königsberg, but because of the war he could only begin to study physics at the age of 23, when he entered the Technical University of Munich. He shortly moved to Göttingen and became a student of Heisenberg. There Symanzik encountered two young colleagues, H. Lehmann and W. Zimmermann, with whom he developed both close friendship and scientific collaboration. This group was later dubbed the "Feldverein" by W. Pauli, when it had become an important influence in theoretical physics.

In 1954, Symanzik completed his doctoral thesis, "On the Schwinger functional in quantum field theory." The deep insights in this work and the technical skill in their implementation set the scene for a series of classic papers in diverse fields of theoretical physics; all these papers share conceptual clarity combined with overwhelming technical ability. The best known work from the period in Göttingen was the famous LSZ "reduction formula" to express scattering cross sections in terms of vacuum expectation values of quantum fields. Today this formula can be found in most books on elementary particles or quantum fields.

From 1955 to 1962, Symanzik worked in many departments in both the United States and in Europe, including the Institute for Advanced Study, the University

of Chicago, Göttingen, Hamburg, Stanford, Princeton, UCLA, and CERN. Two themes during this period were a study of dispersion relations and the analysis of how Green's functions reflect the many-particle structure of quantum fields.

In 1962, Symanzik accepted a professorship at the Courant Institute, where he remained for 6 years. While there he developed Euclidean quantum field theory, surely one of his greatest achievements. He recognized that field theory could be reduced to the structure of classical statistical mechanics. He proposed that integral equations, correlation inequalities, Markovian properties, interacting random paths, and other aspects of classical statistical physics had an interpretation in quantum field theory. Originally Symanzik was motivated by his attempt to solve the existence question for scalar quantum fields by this method, culminating in his 1968 Varenna lectures. Later these ideas led to the reconstruction theorem for quantum theory from Euclidean fields, and they became an integral part of constructive field theory. Ultimately this approach made possible the computations based on high temperature series or computer simulation in lattice gauge theories based on the renormalization group. Furthermore this point of view led to the noninteraction theorems for quartic scalar field theories. Euclidean field theory today is an indispensible starting point for the study of many problems in particle physics.

In 1968, Symanzik returned to Germany as a research Professor at DESY. Here his interests turned in a different direction, and the Callan-Symanzik equation was another high point of his career. This renormalization group equation gave impetus to the discovery of asymptotically free quantum field theories. Symanzik found a first model. Soon thereafter it was recognized that nonabelian gauge theories are asymptotically free. This was a precondition for the development of Quantum Chromodynamics, the currently accepted model for hadronic interactions.

In 1981 the German Physical Society presented Kurt Symanzik the Max Planck Medal, its highest honor for scientific achievement.

For many colleagues and young scientists, Symanzik was a physicist whom one visited in order to learn by conversation. His shyness, his penetrating insight, and his dislike for redundancy in communication often made it difficult to establish personal contact with him. But those who did get to know him closely remember not only an extraordinary intellect, but also a loyal and generous friend. He enjoyed contacts with colleagues and young scientists both at DESY and elsewhere. It was usual for Symanzik to perform long calculations and to write long letters to encourage the work of others as well as to explain his own unique and original insights.

He enjoyed with equal gusto unscientific activities including swimming, attending ballet and dancing. Friends and colleagues watched with amusement and affection as he tried to execute dance steps as complicated as the equations in his papers!

Kurt Symanzik's last papers were devoted to lattice gauge theory. They show that he was in full command of his creative force until the end when he died of cancer on October 25, 1983.

A. Jaffe, H. Lehmann, and G. Mack

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Monte Carlo Simulations for Quantum Field Theories Involving Fermions

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Abstract. We present a new variant of a Monte Carlo procedure for euclidean quantum field theories with fermions. On a lattice every term contributing to the expansion of the fermion determinant is interpreted as a configuration of self-avoiding oriented closed loops which represent the fermionic vacuum fluctuations. These loops are related to Symanzik's polymer description of euclidean quantum field theory. The method is extended to the determination of fermionic Green's functions. We test our method on the Scalapino-Sugar model in one, two, three, and four dimensions. Good agreement with exactly known results is found.

1. Introduction

In recent years Monte Carlo simulations for euclidean lattice models have been of considerable help in improving our understanding of those relativistic quantum field theories, which are supposed to describe high energy particle physics. This includes in particular gauge theories. Now any realistic model for particle interactions includes fermionic fields like quark fields. It is, therefore, important to simulate systems with fermionic degrees of freedom. There have been several proposals to deal with this problem, see e.g. [1–16], or [17–19] for a review. However, all these methods require extensive computing time and some of them only work for two-dimensional models or are only approximations from the beginning: quenched approximation, hopping parameter expansion etc. In particular for the interesting case of four-dimensional lattices no way has yet been found to perform Monte Carlo calculations including fermions as efficiently as they can be done when bosonic fields only are present.

It is the aim of this paper to propose a new numerical method which basically treats all fields on the same footing during the upgrading procedure. This new way of treating fermions applies to all lattice models known to the authors and may

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* On leave of absence from Freie Universität Berlin

easily be combined with the standard methods already used for bosonic fields. We now briefly describe our method, the details of which will be given in the next sections. To deal with the fermionic degrees of freedom we introduce an auxiliary statistical ensemble, the elements of which are labelled by a subset of the set of permutations on the fermionic degrees of freedom. We give a graphical representation of such an element of the statistical ensemble in terms of self-avoiding loops and relate it to Symanzik's polymer description of euclidean quantum field theory [20, 21]. This procedure guarantees that Pauli's exclusion principle for the fermions is automatically taken care of. The antisymmetry in the fermionic degrees of freedom is obtained by equating the Green's functions of the physical model with certain (ratios of) Green's functions of the auxiliary statistical ensemble. This is explained in Sect. 2.

In Sect. 3 we present a local heat bath method for a fermionic action in the form originally introduced by Kogut and Susskind in a Hamiltonian context [22]. As just mentioned, each element of the statistical ensemble, which is a certain permutation, has a graphical representation. The heat bath is local, i.e. most of the fermionic degrees of freedom are frozen, because we apply localized permutations which upgrade a given permutation only locally. The intricate part of our procedure is to show that the principle of detailed balance is satisfied. This combined with the ergodicity of the upgrading ensures that the Gibbs distribution for the auxiliary statistical ensemble is the unique equilibrium distribution for the upgrading procedure.

In Sect. 4 we consider the standard, simple model introduced by Scalapino and Sugar [3], which has often been used to test methods for dealing with fermions numerically. We compare our Monte Carlo results with exact values obtained by numerical Fourier summation. Finally, Sect. 5 contains some conclusions.

2. Statistical Ensembles for Fermions

In this section we will discuss the auxiliary statistical ensemble, in terms of which the fermionic degrees of freedom may be described so as to give the Green's functions of the physical theory. We begin by recalling the standard formulation of lattice theories involving fermions. We will restrict our attention to (finite) cubical lattices in d dimensions. The question of how to choose the right boundary conditions will not be relevant in this section.

Consider a lattice action of the form

$$S = S(\psi^{+}, \psi, \phi). \tag{2.1}$$

ψ^{+} and ψ denote fermionic fields and are considered to be Grassmann variables. ϕ stands for all other fields, which are supposed to be bosonic. In all applications so far, the fermionic fields are defined on the vertices of the lattice. Thus the lattice site x may be used to label the fermionic degrees of freedom. The fields ϕ may live on vertices (ordinary Bose fields) or on links (lattice gauge fields). Our method will also allow for more complicated situations, where for example ϕ lives on higher dimensional cells.

For the purpose of explaining our method, we first look at the simplified situation, where

(a) the fermionic fields carry no indices (like flavour or colour) other than the vertex index,

(b) the action is quadratic in the fermionic fields.

It is easy to extend our method to situations, where these restrictions are removed and at the end of the section we will briefly indicate how this is done. With these restrictions the action S may be decomposed as

$$S(\psi^+, \psi, \phi) = S_F(\psi^+, \psi, \phi) + S_B(\phi), \tag{2.2}$$

where

$$S_F(\psi^+, \psi, \phi) = \sum_{x, y \in \text{Lattice}} \psi^+(x) A_{xy}(\phi) \psi(y). \tag{2.3}$$

Here the matrix $A = A(\phi) = \{A_{xy}(\phi)\}$ with complex valued entries is indexed by the vertices of the lattice and is a functional of the bosonic fields ϕ, and $S_B(\phi)$ is that part of the action which does not involve fermionic fields. The "partition" function of the theory is therefore

$$Z_{FB} = \int d\phi d\psi^+ d\psi e^{-S}. \tag{2.4}$$

The integration over the fermionic fields is in the sense of Berezin [23], and $d\phi$ describes the integration over the bosonic fields ϕ. By

$$\langle X \rangle = Z_{FB}^{-1} \int d\phi d\psi^+ d\psi X e^{-S} \tag{2.5}$$

we denote the expectation value in this model. If we perform the fermionic integration first, Eq. (2.4) may be written as

$$Z_{FB} = \int d\phi \det A(\phi) e^{-S_B(\phi)}. \tag{2.6}$$

To obtain Green's functions involving fermionic fields, we consider the typical example $\langle \psi(v)\psi^+(u) \rangle$. For given u, v let $A^{(u,v)}$ denote the matrix given by

$$A_{xy}^{(u,v)} = \begin{cases} A_{xy} & \text{if} \quad x \neq u \quad \text{and} \quad y \neq v, \\ 1 & \text{if} \quad x = u \quad \text{and} \quad y = v, \\ 0 & \text{otherwise}. \end{cases} \tag{2.7}$$

Then we have

$$\langle \psi(v)\psi^+(u) \rangle = \frac{\int d\phi \det A^{(u,v)}(\phi) e^{-S_B(\phi)}}{\int d\phi \det A(\phi) e^{-S_B(\phi)}}. \tag{2.8}$$

Of course, $\det A^{(u,v)}$ is (up to a sign) equal to the determinant of the submatrix of A obtained by deleting the u^{th} row and v^{th} column. More generally, any (higher order) Green's function in the fermionic fields may be obtained in this way from determinants of suitable submatrices. Formulas (2.6) and (2.8) are obtained from Berezin's integration theory, by using the standard formula

$$\det A = \sum_{\pi} \text{sgn}(\pi) \prod_{x} A_{x\pi(x)}, \tag{2.9}$$

where π runs through the set of all permutations of lattice points.

Let us now neglect the ϕ dependence for a moment. Each term on the right-hand side of (2.9), which is labelled by π, may be given a graphical presentation as follows. If $\pi(x) \neq x$, we draw an oriented straight line in the lattice from the point x to the point $\pi(x)$. By definition its length is 1. Now every permutation may be written as a product of (nontrivial) cyclic permutations

$$\pi = \pi_1 \dots \pi_s, \tag{2.10}$$

and this description is unique up to the ordering of the cyclic factors π_r, $1 \leqq r \leqq s$. If ℓ_r denotes the order of π_r (such that $\ell_r = 2$ if π_r is a transposition), we have

$$\operatorname{sgn}(\pi) = \prod_{r=1}^{s} (-1)^{\ell_r + 1}. \tag{2.11}$$

Using this graphical presentation each cyclic permutation π_r corresponds to a closed oriented polygonal loop $\mathscr{L}_r$ of length ℓ_r. These loops $\mathscr{L}_1, \dots, \mathscr{L}_s$ are nonintersecting in the sense that each vertex in the lattice is the endpoint of at most one oriented straight line and in that case it is also the starting point of exactly one line.

Conversely to each such family of nonintersecting oriented polygonal loops corresponds a unique π and hence a unique contribution to $\det A$ in the sense of Eq. (2.9). Note that we only need to consider those π which are contained in the set

$$\mathscr{C}(A) = \left\{\pi;\ \prod_x A_{x\pi(x)} \neq 0\right\}. \tag{2.12}$$

Similarly, the only nonvanishing contributions to $\det A^{(u,v)}$ are among those π for which $\pi(u) = v$. By deleting the particular straight line going from u to v, each such contribution to $\det A^{(u,v)}$ can be graphically described by a set of nonintersecting loops plus an additional "propagator", i.e. an open polygonal line, going from v to u, which is nonselfintersecting and not intersecting the other loops in the sense just described.

It is important to note that this nonintersecting property is a *local property*, i.e. it is only necessary to test all vertices individually to see whether a given set of polygonal loops correspond to a permutation π or not.

In most applications the matrix A will have the following additional property: A is said to be *local*, if $A_{xy} = 0$ unless $\operatorname{dist}(x, y) \leqq 1$. In the context of our graphical presentation this means that all loops are built out of links. Also the nonintersecting property stated above is now the usual nonintersecting property of curves. For local A, we say $\pi \in \mathscr{C}(A)$ is a "dimer" if π is cyclic of order 2. π is said to contain a dimer if at least one of its cyclic factors π_r is of order two. Now for local A, the number $|\mathscr{C}(A)|$ of elements in $\mathscr{C}(A)$ is bounded above by $(2d+1)^{(\text{volume of lattice})}$. On the other hand, since the density of states for the dimer problem is explicitly known for $d = 2$ at zero temperature [24–27], it is easy to obtain the lower bound $\alpha^{\text{volume}/2}$ on $|\mathscr{C}(A)|$ $\left(\alpha = \exp\frac{2G}{\pi} = 1.791\dots,\ G = \text{Catalan's} = 0.915\dots\right)$, if A is such that $A_{xy} \neq 0$ whenever $\operatorname{dist}(x, y) \leqq 1$. Indeed, this follows by considering the subset of $\mathscr{C}(A)$ consisting of those π with dimers only which point in directions parallel to a given plane. In Sect. 3 we will also give a discussion on how to compute $|\mathscr{C}(A)|$ numerically for the case of such a local A.

The graphical presentation we have given for general (constant) matrix A is related to Symanzik's polymer presentation [20, 21], see also [28, 29], as follows. For simplicity, let A be of the special form

$$A = \mathbb{1} + \Gamma, \tag{2.13}$$

with $\Gamma_{xx}=0$ and Γ_{xy} small. Then

$$\begin{aligned}\det A &= \exp[\operatorname{tr}\ln A] \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr}\Gamma^n\right].\end{aligned} \tag{2.14}$$

Since

$$\operatorname{tr}\Gamma^n = \sum_{x_1,\ldots,x_n} \Gamma_{x_1x_2}\ldots\Gamma_{x_nx_1}. \tag{2.15}$$

each term of the right-hand side of (2.15) can be viewed graphically as an oriented polygonal loop of length n. In this formulation, however, there is no nonintersecting condition, i.e. each vertex may be the endpoint of more than one oriented straight line. The converse holds also in this case: To each such loop of length n corresponds a unique contribution to $\operatorname{tr}\Gamma^n$ via (2.15). The Taylor expansion of the exponent in Eq. (2.14) then gives a combinatorial description (and proof) of how these terms in (2.14)–(2.15), each of which is presented by one oriented loop, combine to the set of terms in (2.9), each of which is presented by a family of nonintersecting oriented loops. A similar graphically equivalent description may be given for the Green's functions, e.g. $\langle\psi(v)\psi^+(u)\rangle$.

We now turn to a construction of the statistical ensemble. Let the matrix $|A|$ be given by

$$|A|_{xy} = |A_{xy}|, \tag{2.16}$$

and consider the permanent of $|A|$,

$$\operatorname{Per}(|A|) = \sum_{\pi\in\mathscr{C}(A)} \prod_x |A|_{x\pi(x)}. \tag{2.17}$$

Now $\mathscr{C}(A)$ may be viewed as a Gibbs statistical ensemble with energies given by

$$E(\pi, A) = \sum_x E_x(\pi, A) = -\sum_x \ln(|A|_{x\pi(x)}), \tag{2.18}$$

such that the permanent becomes the corresponding partition function

$$\operatorname{Per}(|A|) = \sum_{\pi\in\mathscr{C}(A)} \exp[-E(\pi, A)]. \tag{2.19}$$

If we denote by

$$\bar{X} = \frac{1}{\operatorname{Per}(|A|)} \cdot \sum_\pi X(\pi)\cdot e^{-E(\pi,A)} \tag{2.20}$$

averages in this statistical ensemble, we have the following relation for the determinant of A

$$\det A = \sum_{\pi} \operatorname{sgn}(\pi, A) e^{-E(\pi, A)} = \operatorname{Per}(|A|) \cdot \overline{\operatorname{sgn}(A)}, \tag{2.21}$$

where $\operatorname{sgn}(\pi, A)$ is defined on $\mathscr{C}(A)$ by

$$\operatorname{sgn}(\pi, A) = \operatorname{sgn}(\pi) \cdot \prod_x \frac{A_{x\pi(x)}}{|A_{x\pi(x)}|}. \tag{2.22}$$

To obtain the Green's function $\langle \psi(v)\psi^+(u)\rangle$ we have to determine [cf. Eq. (2.8)]

$$\det A^{(u,v)} = \operatorname{Per}(|A^{(u,v)}|) \cdot \overline{\operatorname{sgn}(A^{(u,v)})}, \tag{2.23}$$

where now the configurations of $\mathscr{C}(A^{(u,v)})$ have to be taken into account.

We remark that the following modification of this construction leads to the theory of the noninteracting polymer gas: Take A to be local with $A_{xx} = 1$ and $A_{xy} = \text{const}$ for $\operatorname{dist}(x, y) = 1$. The statistical ensemble is defined to be the subset of $\mathscr{C}(A)$ consisting of all π's without dimers. In fact, the numerical results in [42] were obtained by a corresponding modification of the upgrading procedure, to be explained in Sect. 3.

Let us return to the case, where bosonic fields ϕ are present. Again we define $\mathscr{C}(A)$ by (2.12), where the condition $\prod_x A_{x\pi(x)} \neq 0$ is now understood in the sense of functionals. Also in (2.18) we set $E_x(\pi, A(\phi)) = \infty$ for any value of ϕ for which $A_{x\pi(x)}(\phi) = 0$. We are now in a position to describe the auxiliary statistical ensemble and express physical Green's functions as expectation values in this theory. In fact, the thermal average values are now

$$\bar{X} = Z^{-1} \int d\phi \sum_{\pi \in \mathscr{C}(A)} X(\pi, \phi) e^{-S_B(\phi) - E(\pi, A(\phi))}, \tag{2.24}$$

with

$$Z = \int d\phi \sum_{\pi \in \mathscr{C}(A)} e^{-S_B(\phi) - E(\pi, A(\phi))}. \tag{2.25}$$

The set of configurations is now the product $\mathscr{C}(A)$ times the set of the usual bosonic configurations. Consider first an observable F which only depends on ϕ. Then

$$\begin{aligned}\langle F(\phi)\rangle &= \frac{\int d\phi F(\phi) \det A(\phi) e^{-S_B(\phi)}}{\int d\phi \det A(\phi) e^{-S_B(\phi)}} \\ &= \frac{\int d\phi \sum_{\pi \in \mathscr{C}(A)} F(\phi) \operatorname{sgn}(\pi, A(\phi)) e^{-S_B(\phi) - E(\pi, A(\phi))}}{\int d\phi \sum_{\pi \in \mathscr{C}(A)} \operatorname{sgn}(\pi, A(\phi)) e^{-S_B(\phi) - E(\pi, A(\phi))}} \\ &= \overline{F \cdot \operatorname{sgn}(A)} / \overline{\operatorname{sgn}(A)}.\end{aligned} \tag{2.26}$$

We note a crucial property of this relation, which will allow us to perform *local upgrading procedures* both for the fermionic and the bosonic part: $E(\pi, A(\phi))$ is a sum of local terms $E_x(\pi, A(\phi))$ in the sense that for local theories the latter will depend only on the form of π near x and on the field configurations of ϕ which live near x.

Next we turn to fermionic Green's functions. For the two-point function we obtain as a generalization of Eq. (2.23) [see Eq. (2.8)]

$$\langle \psi(v)\psi^+(u)\rangle = \frac{\overline{\text{sgn}(A^{(u,v)})}}{\overline{\text{sgn}(A)}}. \tag{2.27}$$

Analogously higher order Green's functions may be obtained by augmenting the statistical ensemble appropriately.

The remainder of this section is devoted to a brief outline of the modifications necessary to cover the cases that

(a′) the fermionic fields carry indices ϱ which may include the vertex index x and internal degrees of freedom and

(b′) higher order interactions in the fermionic fields are present.

In the case (a′), which is important for treating e.g. non-abelian lattice gauge theories, A is a matrix $A_{\varrho x}$. And $\mathscr{C}(A)$ is now defined to be the set of all permutations π of the ϱ's for which $\prod_\pi A_{\varrho\pi(\varrho)}(\phi) \neq 0$. Again there is a graphical representation which now is in d' dimensions $(d' > d)$ with the extra $(d'-d)$ dimensions being used to describe the additional degrees of freedom.

As for the case (b′) assume the action contains an extra term which is of fourth order in the fermionic fields

$$S'_F(\psi^+, \psi, \phi) = \sum_{\substack{\varrho \neq \varrho' \\ \varkappa \neq \varkappa'}} \psi^+(\varrho)\psi^+(\varrho')B_{\varrho\varrho'\varkappa\varkappa'}\psi(\varkappa)\psi(\varkappa'). \tag{2.28}$$

Let $\mathscr{I} = (I, I_1, \ldots, I_k)$ $(0 \leq k)$ be a decomposition of the set of all ϱ's into a subset I and ordered sets I_ℓ $(1 \leq \ell \leq k)$ containing two elements each. Then

$$\begin{aligned}&\int d\psi^+ d\psi e^{-S_F(\psi^+, \psi, \phi) - S'_F(\psi^+, \psi, \phi)} \\ &= \sum_{\pi, \mathscr{I}} \text{sgn}(\pi) \prod_{\varrho \in I} A_{\varrho\pi(\varrho)}(\phi) \prod_{\substack{\ell=1 \\ (\varrho', \varrho'') = I_\ell}}^{k} B_{\varrho'\varrho''\pi(\varrho')\pi(\varrho'')}(\phi).\end{aligned} \tag{2.29}$$

The auxiliary statistical ensemble is now labelled by $\pi, \mathscr{I} \in \mathscr{C}(A, B)$, and the states representing the bosonic fields ϕ, where

$$\mathscr{C}(A, B) = \left\{ (\pi, \mathscr{I});\ \prod_{\varrho \in I} A_{\varrho\pi(\varrho)} \prod_{\substack{\ell=1 \\ (\varrho, \varrho') = I_\ell}}^{k} B_{\varrho\varrho'\pi(\varrho)\pi(\varrho')} \neq 0 \right\}. \tag{2.30}$$

Also the energies (2.18) are replaced by

$$E(\pi, \mathscr{I}, A(\phi), B(\phi)) = -\sum_{\varrho \in I} \ln|A_{\varrho\pi(\varrho)}(\phi)| - \sum_{\substack{\ell=1 \\ (\varrho, \varrho') = I_\ell}}^{k} \ln|B_{\varrho\varrho'\pi(\varrho)\pi(\varrho')}(\phi)|. \tag{2.31}$$

Again a graphical presentation may be obtained. It is obvious how to extend this procedure to interactions of order higher than four.

Another strategy would work for local and translation invariant B's. One can introduce an intermediate boson $\varphi(\varrho, \varkappa)$ and replace (2.28) by

$$S_F''(\psi^+, \psi, \varphi, \phi) = -\sum [\varphi(\varrho, \varkappa) B_{\varrho\varrho'\varkappa\varkappa'} \varphi(\varrho', \varkappa') + \psi^+(\varrho)\psi(\varkappa)(B_{\varrho\varrho'\varkappa\varkappa'} + B_{\varrho'\varrho\varkappa'\varkappa})\varphi(\varrho', \varkappa')], \quad (2.32)$$

which again turns out to be an action of case (a').

3. The Heat Bath Method

In this section we explain our heat bath method for the determination of fermion determinants for the case of free massive fermions.

There exist different formulations for fermions on a lattice, e.g. the Wilson [30], Kogut-Susskind [31–36], and Dirac-Kähler [37–39] versions. For the purpose of this paper the second version is the most adequate one. Thus we use it to exemplify our heat bath method in spite of its shortcoming, namely numerical results are reliable only for sufficiently heavy fermions. More precisely, the statistical errors obscure the measurements if the hopping parameter $k = 1/(2am)$ (a = lattice spacing, m = fermion mass) exceeds 0.6, 0.25, 0.15 for Susskind fermions in 2, 3, and 4 dimensions, respectively. In a forthcoming publication we shall extend our method to the case of Wilson fermions, where there is no such restriction on the mass.

We consider a hypercubic lattice in d dimensions with lattice points

$$\mathbf{x} = \sum_{\mu=1}^{d} x_\mu \mathbf{a}_\mu \quad (3.1)$$

labelled by integer components, $x_\mu = 0, 1, 2, \ldots, L-1$, and $\{\mathbf{a}_\mu\}$ orthogonal vectors parallel to the lattice axes, $\mathbf{a}_\mu \cdot \mathbf{a}_\nu = a^2 \delta_{\mu\nu}$. The naive lattice version of the euclidean free fermion action reads

$$S = \sum_x \psi^+(x)(\gamma_\mu \partial_\mu + m)\psi(x) \quad (3.2)$$

with central differences, i.e.

$$\partial_\mu \psi(x) = \frac{1}{2a}[\psi(x + a_\mu) - \psi(x - a_\mu)].$$

The Susskind formulation is most easily obtained by the transformation [35, 36]

$$\psi(x) \to \prod_{\mu=1}^{d} \gamma_\mu^{x_\mu} \psi(x); \quad (3.3a)$$

the γ-matrices then become proportional to the unit matrix, albeit x-dependent,

$$\gamma_\mu \to \gamma_\mu(x)\mathbb{1} = \prod_{\nu=1}^{\mu-1} (-1)^{x_\nu} \mathbb{1}. \quad (3.3b)$$

It is then sufficient to consider only one component of a full Dirac fermion at each lattice site, and the matrix which appears in Eq. (2.3) takes the form

$$(\gamma_\mu \partial_\mu + m)_{\mathbf{x},\mathbf{y}} = \gamma_\mu(x) \frac{1}{2a}[\delta_{\mathbf{x}+\mathbf{a}_\mu,\mathbf{y}} - \delta_{\mathbf{x}-\mathbf{a}_\mu,\mathbf{y}}] + m\delta_{\mathbf{x},\mathbf{y}}. \quad (3.4)$$