ALM 16

Advanced Lectures in Mathematics

Transformation Groups and Moduli Spaces of Curves

变换群与曲线模空间

Editors: Lizhen Ji · Shing-Tung Yau

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Bianhuanqun Yu Quxian Mokongjian

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Preface

Transformation groups have played a fundamental role in many areas of mathematics such as differential geometry, geometric topology, algebraic topology, algebraic geometry, number theory. One of the basic reasons for their importance is that symmetries are described by groups (or rather group actions). Indeed, the existence of group actions makes the spaces under study more interesting, and properties of groups can also be understood better by studying their actions on suitable spaces.

Quotients of smooth manifolds by group actions are usually not smooth manifolds. On the other hand, if the actions of the groups are proper, then the quotients are orbifolds.

The notion of V-manifolds was first introduced by Satake in 1956 in the context of locally symmetric spaces and automorphic forms. V-manifolds were reintroduced and renamed orbifolds by Thurston near the end of 1978 in connection with the Thurston geometrization conjecture on the geometry of three dimensional manifolds. Basically, orbifolds are locally quotients of smooth manifolds by finite groups. Besides arising from transformation groups, many natural spaces in number theory and algebraic geometry are orbifolds.

An important example of such interaction is given by the action of the mapping class groups on the Teichmüller spaces, and the quotients give the moduli spaces of Riemann surfaces (or algebraic curves) and are orbifolds. One reason for the importance of this group action is that Riemann surfaces are fundamental objects in complex analysis, differential and complex geometry, low dimensional topology, algebraic geometry, number theory, mathematical physics etc., and the Teichmüller spaces are moduli spaces of marked Riemann surfaces. These moduli spaces and their variants have played a fundamental role in algebraic geometry and string theory. Properties of the moduli spaces can sometimes be understood more easily through this action on the Teichmüller spaces.

The moduli spaces of algebraic curves are noncompact and admit a well-known compactification, called the Deligne-Mumford compactification. An important fact is that the Deligne-Mumford compactification is also a compact orbifold.

The above discussions show that orbifolds arise naturally from different contexts. Recently, orbifolds have also found striking applications in algebraic geometry and string theory such as the McKay correspondence.

To introduce these basic and important concepts to the younger generation, two consecutive summer schools were organized at the Center of Mathematical Sciences, Zhejiang University: *Transformation Groups and Orbifolds* from June

30 to July 11, 2008, and *Geometry of Teichmüller Spaces and Moduli Spaces of Curves* from July 14 to July 20, 2008. Experts on topics related to transformation groups, orbifolds, Teichmüller spaces, mapping class groups, and moduli spaces of curves were invited to give expository lecture series. This book contains the expanded lecture notes of some of these lecture series.[1]

We would like to thank the speakers for their hard work in preparing the talks and writing up the lecture notes, and the referees for carefully reading the lecture notes and making valuable suggestions and comments. We hope that this book will convey the lively spirit and freshness of the lectures at the summer schools, and believe that it will be a valuable source for people who want to learn these beautiful topics.

Lizhen Ji
Shing-Tung Yau
January 22, 2010

[1]The last lecture of C.C. Liu is related to the paper Formulae of one-partition and two-partition Hodge integrals, *Geometry & Topology Monographs* 8 (2006) 105–128. We would like to thank the editors of the Geometry & Topology Monographs for their permission to allow us to reprint this paper here.

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Lectures on Orbifolds and Group Cohomology*

Alejandro Adem† and Michele Klaus‡

Abstract

The topics discussed in these notes include basic properties and definitions of orbifolds, and aspects of their cohomology and K–theory. Connections to group cohomology and equivariant algebraic topology appear in the context of orbifolds and their associated invariants. These notes are based on lectures given by the first author at the summer school on *Transformation Groups and Orbifolds*, held at Hangzhou, China in June/July 2008.

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1 Introduction

Orbifolds and their invariants play an important role in mathematics. The study of basic examples of quotients by Lie groups acting with finite isotropy on smooth compact manifolds leads to applications of ideas and techniques ranging from differential geometry and topology to algebraic geometry, group cohomology, homotopy theory and mathematical physics.

In these lecture notes we present some basic definitions and properties of orbifolds emphasizing their connections to algebraic topology and group cohomology. The language of groupoids provides a convenient mechanism for connecting these apparently distinct topics, and the global perspective this provides yields useful insight. In particular techniques from classical transformation groups can be used to construct interesting examples and formulate calculations in terms of better understood invariants from algebraic topology, such as cohomology and K–theory. Plenty of examples are provided both as a source of motivation and as a way to facilitate the understanding of the theory. We also discuss a *stringy* product in orbifold K–theory that was recently introduced in [5], which is motivated by the Chen–Ruan product in orbifold cohomology.

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These notes are intended for graduate students interested in the general topic of orbifolds and their invariants. They reproduce the lectures given at the *2008 Hangzhou Summer School on Orbifolds and Transformation Groups* by the first author. Thus they are not meant to be complete or fully rigorous; rather their goal is to motivate casual readers to learn more about the subjects discussed here by consulting the literature; we offer the book [2] and the references therein as a good place to start. Moreover for these notes this book will be the standard reference, and we will omit referring to it to avoid repetition.

Both authors would like to thank the organizers of the summer school for their hospitality, and in particular Lizhen Ji for his wonderful enthusiasm for mathematics and his encouragement to write up these notes.

2 Classical orbifolds

In this section we give a definition of an orbifold from a geometric point of view which is close to the original one (see [23] for Satake's definition of V-manifold).

Definition 2.1. Let X be a topological space and fix $n \geqslant 0$.

(1) An n-dimensional *orbifold chart* on X is given by a connected open subset $\widetilde{U} \subseteq \mathbb{R}^n$, a finite group G of effective smooth automorphisms of $\widetilde{U}$, and a map $\varphi : \widetilde{U} \to X$ such that φ is G-invariant and induces an homeomorphism of $\widetilde{U}/G$ onto an open subset $U \subseteq X$.

(2) An *embedding* $\lambda : (\widetilde{U}, G, \varphi) \to (\widetilde{V}, H, \psi)$ between two charts is a smooth embedding $\lambda : \widetilde{U} \to \widetilde{V}$ such that $\psi \circ \lambda = \varphi$.

(3) An *orbifold atlas* on X is a family $\mathfrak{U} = \left\{(\widetilde{U}, G, \varphi)\right\}$ of charts which cover X and are locally compatible: given two charts $(\widetilde{U}, G, \varphi)$ with $U = \varphi(\widetilde{U})$ and $(\widetilde{V}, H, \psi)$ with $V = \psi(\widetilde{V})$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ of x and a chart $(\widetilde{W}, K, \phi)$ with $\phi(\widetilde{W}) = W$ and such that there are two embeddings $\lambda : (\widetilde{W}, K, \phi) \to (\widetilde{V}, H, \psi)$ and $\mu : (\widetilde{W}, K, \phi) \to (\widetilde{U}, G, \varphi)$.

(4) An atlas $\mathfrak{U}$ *refines* another atlas $\mathfrak{W}$ if for every chart in $\mathfrak{U}$ there exists an embedding into some chart of $\mathfrak{W}$. Two orbifold atlases are *equivalent* if they have a common refinement.

Definition 2.2. A (*classical*) *orbifold* $\mathfrak{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathfrak{U}]$ of n-dimensional orbifold atlases.

Remark. We collect here some technical facts about orbifolds that are supposed to give a better understanding of the above definition:

(1) For every chart $(\widetilde{U}, G, \varphi)$ of an orbifold $\mathfrak{X}$, the group G acts freely on a dense open subset of $\widetilde{U}$.

(2) By local smoothness, every orbifold has an atlas consisting of linear charts $(\mathbb{R}^n, G, \varphi)$ where $G \subset O(n)$ (see [9]).

(3) An embedding $\lambda : (\widetilde{U}, G, \varphi) \to (\widetilde{V}, H, \psi)$ between two charts induces an injection $\lambda : G \to H$.
(4) Every atlas is contained in a unique maximal one and two atlases are equivalent if and only if they are contained in the same maximal one.
(5) If all the G-actions of an atlas are free, then $\mathfrak{X}$ is an honest manifold.

Given the remarks above, we can think of an orbifold as a "space with isolated singularities"; a notion that we make more precise with the next two definitions:

Definition 2.3. Let $x \in X$ with $\mathfrak{X} = (X, \mathfrak{U})$ an orbifold. The *local group* at x is the group $G_x = \{g \in G | gu = u\}$ where $(\widetilde{U}, G, \varphi)$ is any local chart with $\varphi(u) = x$. The group G_x is well defined up to conjugation. For an orbifold $\mathfrak{X} = (X, \mathfrak{U})$ its *singular set* is the subspace $\Sigma(\mathfrak{X}) = \{x \in X | G_x \neq 0\}$. A point in $\Sigma(\mathfrak{X})$ is a *singular point* of the orbifold $\mathfrak{X}$.

Let us now turn our attention to the notion of a map between two orbifolds (which turns out to be a more subtle concept that one might expect, as we will see later). We give a first definition in the current geometric setting:

Definition 2.4. Let $\mathfrak{X} = (X, \mathfrak{U})$ and $\mathfrak{Y} = (Y, \mathfrak{V})$ be two orbifolds. A map $f : X \to Y$ is a *smooth map between orbifolds* if for any point $x \in X$ there are charts $(\widetilde{U}, G, \varphi)$ around x and $(\widetilde{V}, H, \psi)$ around $f(x)$, with the property that f maps $\varphi(\widetilde{U})$ into $\psi(\widetilde{V})$ and can be lifted to a smooth map $\widetilde{f} : \widetilde{U} \to \widetilde{V}$ with $\psi\widetilde{f} = f\varphi$. Two orbifolds are *diffeomorphic* if there are smooth maps $f : X \to Y$ and $g : Y \to X$ with $fg = Id_Y$ and $gf = Id_X$.

A way to construct orbifolds is to take the quotient of a manifold by some nice group action. Let M be a smooth manifold and G a compact Lie group acting smoothly, effectively and almost freely on M (i.e. with finite isotropy). For each element $x \in M$ there is a chart $U \cong \mathbb{R}^n$ of M around x which is G_x invariant. The triples $(U, G_x, \pi : U \to U/G_x)$ are the orbifold charts.

Definition 2.5. A *quotient orbifold* is an orbifold given as the quotient of a smooth, effective, almost free action of a compact Lie group G on a smooth manifold M. If the group G is finite, the associated orbifold is called a *global quotient.*

Remark. If a compact Lie group G acts smoothly and almost freely on a manifold M, then we have a group extension:

$$1 \to G_0 \to G \to G_{eff} \to 1$$

where G_0 is finite and G_{eff} acts effectively. Even though $M/G = M/G_{eff}$, the original G-action does not give a classical orbifold. This will be one of the motivations for a more general definition of an orbifold, not involving the effective condition (see Definition 5.15).

3 Examples of orbifolds

(a) Consider a finite subgroup $G \subset GL_n(\mathbb{Z})$; it acts smoothly on the torus $X = \mathbb{R}^n/\mathbb{Z}^n$ giving rise to a so called *toroidal orbifold* $X \to X/G$ (see [3] for a discussion of their properties). Many important examples are of this form.

• The matrix $-I \in GL_4(\mathbb{Z})$ defines a $\mathbb{Z}/2$-action given by $\tau(z_1, z_2, z_3, z_4) = (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1})$. The quotient $\mathbb{T}^4/G$ is the Kummer surface and it has sixteen isolated singularities.

• The group $\mathbb{Z}/4$ acts on $\mathbb{C}^3$ via $\tau(z_1, z_2, z_3) = (-z_1, iz_2, iz_3)$. There is a lattice $M \subset \mathbb{C}^3$ on which the action has the form:

$$\tau(a_1, a_2, a_3, a_4, a_5, a_6) = (a_1^{-1}, a_2^{-1}, a_4, a_3^{-1}, a_6, a_5^{-1}).$$

This gives rise to a $\mathbb{Z}/4$-action on $\mathbb{T}^6$ which has 16 isolated fixed points and $[\mathbb{T}^6]^{\mathbb{Z}/2}$ consists of 16 copies of $\mathbb{T}^2$. This example arises in the work of Vafa and Witten and has also been studied by Joyce who has shown that it has 5 different desingularizations (see [16]).

• The action of $(\mathbb{Z}/2)^2$ on $\mathbb{T}^6$ defined on generators by:

$$\sigma_1(z_1, z_2, z_3, z_4, z_5, z_6) = (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}, z_5, z_6)$$
$$\sigma_2(z_1, z_2, z_3, z_4, z_5, z_6) = (z_1^{-1}, z_2^{-1}, z_3, z_4, z_5^{-1}, z_6^{-1})$$

defines a toroidal orbifold $\mathbb{T}^6/(\mathbb{Z}/2)^2$ with $\{\pm 1\}^6$ as the set of fixed points and $(\mathbb{T}^6)^{<\sigma_1>} \cong (\mathbb{T}^6)^{<\sigma_2>} \cong \mathbb{T}^2 \times \{\pm 1\}^4$. Joyce showed that in contrast to the previous example, this orbifold has many desingularizations (see [16]).

(b) There are also beautiful examples defined using algebraic equations. Let Y be the degree 5 hypersurface of $\mathbb{C}P^4$ defined by the equation:

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \varphi z_0 z_1 z_2 z_3 z_4 = 0$$

where φ is a generic constant. The group $G = (\mathbb{Z}/5)^3$ acts on Y via:

$$e_1(z_0, z_1, z_2, z_3, z_4) = (\lambda z_0, z_1, z_2, z_3, \lambda^{-1} z_4)$$
$$e_2(z_0, z_1, z_2, z_3, z_4) = (z_0, \lambda z_1, z_2, z_3, \lambda^{-1} z_4)$$
$$e_3(z_0, z_1, z_2, z_3, z_4) = (z_0, z_1, \lambda z_2, z_3, \lambda^{-1} z_4)$$

where λ is a fifth root of the unity and the e_i's are the obvious generators of G. The quotient Y/G is the very popular *mirror quintic.*

(c) Another family of examples arises from the natural action of the permutation group S_n on the product $M^n = M \times \cdots \times M$ of n copies of a smooth manifold M. The quotient space $SP^n(M) = M^n/S_n$ is called the symmetric product and is of great interest in algebraic geometry and topology.

(d) Yet another family of important examples arises from quotient singularities of the form $\mathbb{C}^n/G$ for a subgroup $G \subset GL_n(\mathbb{C})$. They have the structure of an algebraic variety arising from the algebra of G-invariant polynomials in $\mathbb{C}^n$. They appear in the context of the McKay correspondence (see [22]).

(e) For a choice of $n+1$ coprime integers $a_0, ..., a_n$; the circle group S^1 acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ as follows:

$$\lambda(z_0, ..., z_n) = (\lambda^{a_0} z_0, ..., \lambda^{a_n} z_n)$$

for every $\lambda \in S^1$. Since the integers are coprime, the action is effective and the quotient orbifold $W\mathbb{P}(a_0, ..., a_n) = S^{2n+1}/S^1$ is called the *weighted projective space*. The case $W\mathbb{P}(1,2)$ has the shape of a teardrop. The $W\mathbb{P}$'s are examples of orbifolds which are NOT global quotients.

4 Orbifolds and manifolds

Similarly to the case of manifolds, we can construct a tangent bundle over an orbifold. The tangent bundle of an orbifold carries the following properties reminiscent of the manifold structure:

Proposition 4.1. *The tangent bundle $T\mathfrak{X} = (TX, T\mathfrak{U})$ of an n-dimensional orbifold has the structure of a $2n$-dimensional orbifold and the projection $p : T\mathfrak{X} \to \mathfrak{X}$ defines a smooth map of orbifolds with fibers $p^{-1}(x) = T_{\tilde{x}}\widetilde{U}/G_{\tilde{x}}$.*

Remark. The tangent bundle is an important object because it allows us to define some of the manifold structures over orbifolds. We can construct for example the dual bundle $T^*\mathfrak{X}$ of $T\mathfrak{X}$, the frame bundle $Fr(\mathfrak{X})$ and the exterior power $\bigwedge T^*\mathfrak{X}$. In this way we can also define Riemannian metrics, almost complex structures, orientability, differential forms and De Rham cohomology.

The objects above satisfy, among others, the following properties:

Proposition 4.2. *The orbifold De Rham cohomology with real coefficients depends only on the underlying space, i.e $H^*_{DR}(\mathfrak{X}, \mathbb{R}) \cong H^*(X, \mathbb{R})$. If $\mathfrak{X}$ is an orientable orbifold, then $H^*_{DR}(\mathfrak{X})$ is a Poincaré duality algebra, in particular for a proper, almost free action of a compact Lie group G on a smooth manifold M, the De Rham cohomology of the quotient orbifold satisfies Poincaré duality.*

This in particular says that De Rham cohomology is not the most appropriate for orbifolds since, in the case of a group action on a manifold for example, it only carries information about the quotient, forgetting the group action giving rise to it. Another interesting result is:

Theorem 4.3. *For a given orbifold $\mathfrak{X}$, its frame bundle $Fr(\mathfrak{X})$ is a smooth manifold with a smooth, effective and almost free $O(n)$-action. In this way $\mathfrak{X}$ is naturally isomorphic to the resulting quotient orbifold $Fr(\mathfrak{X})/O(n)$.*

Remark. The theorem above says in particular that every classical orbifold is a quotient orbifold. The manifold and the group action from which we can obtain a given orbifold are not unique.

5 Orbifolds and groupoids

We now set some categorical notions which will be used to re-define and generalize the concept of an orbifold. Our work will be justified by Theorem 5.14 (see [21]).

Definition 5.1. A *groupoid* is a small category in which every morphism is an isomorphism.

Definition 5.2. A *topological groupoid* $\mathfrak{G}$ is a groupoid whose sets of objects G_0 and arrows G_1 are endowed with a topology in such a way that the five following maps are continuous:

(1) $s : G_1 \to G_0$, where $s(g)$ is the source of g,
(2) $t : G_1 \to G_0$, where $t(g)$ is the target of g,
(3) $m : G_{1\,s}\times_t G_1 = \{(h, g) \in G_1 \times G_1 | s(h) = t(g)\} \to G_1$, where $m(h, g) = h \circ g$ is the composition,
(4) $u : G_0 \to G_1$, where $u(x)$ is the identity of x,
(5) $i : G_1 \to G_1$, where $i(g)$ is the inverse of g.

Definition 5.3. A *Lie groupoid* $\mathfrak{G}$ is a topological groupoid where G_0 and G_1 are smooth manifolds with s, t smooth submersions and m, u and i smooth maps.

Example 5.4. Let G be Lie group acting smoothly from the left on a smooth manifold M. One defines a Lie groupoid $G \ltimes M$ by setting $(G \ltimes M)_0 = M$ and $(G \ltimes M)_1 = G \times M$. The source map $s : G \times M \to M$ is the projection, the target map $t : G \times M \to M$ is the action and the composition is defined by the product in the group G: if $(g, x) \in G \times M$ and $(g', x') \in G \times M$ are such that $s(g, x) = x = g'x' = t(g', x')$ then $(g, x) \circ (g', x') = (gg', x')$.

Definition 5.5. Let $\mathfrak{G}$ be a Lie groupoid. For a point $x \in G_0$ the set of all arrows from x to x form group denoted by G_x and called the *isotropy group* at x. The set $ts^{-1}(x)$ of targets or arrows out of x is called the *orbit* of x. The *orbit space* $|\mathfrak{G}|$ of $\mathfrak{G}$ is the quotient space of G_0 under the relation: $x \sim y$ iff x and y are in the same orbit, i.e. iff there is an arrow going from x to y.

Remark. Since $G_x = s^{-1}(x) \cap t^{-1}(x)$ and s and t are submersions, we have that G_x is a Lie group.

Before establishing the connection between orbifolds and groupoids, we need more definitions:

Definition 5.6. Let $\mathfrak{G}$ be a Lie groupoid.

(1) $\mathfrak{G}$ is *proper* if $(s, t) : G_1 \to G_0 \times G_0$ is proper (recall that a map is proper if the pre-image of every compact is compact),
(2) $\mathfrak{G}$ is a *foliation groupoid* if each isotropy group is discrete,
(3) $\mathfrak{G}$ is *étale* if s and t are local diffeomorphisms. In this case we define the *dimension* of $\mathfrak{G}$ as follow: $\dim(\mathfrak{G}) = \dim(G_0) = \dim(G_1)$.

We remark immediately that every étale groupoid is a foliation groupoid. Furthermore if $\mathfrak{G}$ is a proper foliation groupoid, then all the isotropy groups are