MANUAL OF
MATHEMATICAL PHYSICS

by

PAUL I. RICHARDS

Technical Operations, Inc., Burlington, Massachusetts, U.S.A

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PREFACE

THE goal of this book is to condense established theoretical physics, its applications and its mathematical equipment into a single reference volume of reasonable size without sacrificing either logical continuity or fundamentals. In this way, each formula appears in its deductive context and its origin, as well as any approximations or assumptions which it may entail, can readily be determined.

To render this ideal more approachable the Physics section has been limited to theories which have been well established by experiment, and their deductive ramifications have been terminated while the results still maintain a wide utility in applications. Likewise, the Mathematics section has been economized by omitting much material which can be found in tables of integrals or in compilations of the properties of the classical functions of analysis.

To facilitate rapid extraction of information, an attempt was made either to define or to cross-reference every special concept and every symbol within at least a few pages preceding its every appearance, although some exceptions necessarily occur with the more standard symbols.

Three guides for finding information have been provided: (1) The table of contents displays the overall organization of the material and lists the major subjects within each chapter. (The chapter number forms the first half of each equation number.) (2) The left-hand page headings designate the major subject area, while the right-hand page headings mention individual items which appear on the corresponding pair of pages. (3) The index is as complete as the author could make it and should suffice to locate any item contained in the book.

The index has also been designed for use as a dictionary of terms and concepts, by including even items which are merely mentioned or parenthetically defined in the text.

Any suggestions or comments which users may feel would add to the general utility of the book, either as a reference or as a study guide, will be gratefully received.

CONTENTS

PART 1

PHYSICS

PHILOSOPHY

UNLIKE mathematics, physics has as its prime purpose the description of the real world. The mathematician need only assure himself that his theorems follow from his axioms, but the physicist must continually ask Nature to pass judgement on the smallest facet of his theories. The *facts* of the world alone are the final judges of the usefulness and truth of any theoretical model. Ideally, there should be no disagreement whatever, but we must frequently make do with imperfect theories, being fully cognizant of their limitations.

Yet physics (especially of all the sciences) is not a mere listing of data. Rather physics is mainly a set of mathematical models of Nature. These models serve the dual purpose of describing the logical interrelations of various facts and at the same time of summarizing great masses of data *which can then be discarded.*

Unlike the mathematician, the physicist must stand ready at all times to abandon previously "established" theories. However many and varied may be the experiments which have "verified" a theory, it has never been proved; a new fact of Nature can always arise to contradict it. Yet "well established" theories are never wholly wrong; if they properly describe a wide range of data, they must be good approximations and, indeed, must appear as special cases of any more general theory. Thus Newtonian Mechanics, while now known to be wrong in a philosophic sense, remains a respected and indispensable tool of physics. Under appropriate conditions, it gives to high accuracy the same answers as the more ponderous and inconvenient theories of Quantum Mechanics or of Relativity, of each of which it is a special case. Not to use it would be mere pedantry.

CHAPTER 1

MECHANICS

A. POINT MASSES AND RIGID BODIES

Point Particles; Fundamental Concepts

DENOTE the position of a "small" particle (mass point) by the vector, $\mathbf{r}$, relative to any convenient origin.

It is found experimentally that the acceleration, $d^2\mathbf{r}/dt^2$, of a mass point is, in many circumstances, independent of its previous motion and depends only on its position (and sometimes on the time). Moreover, different mass points are usually found to suffer proportional accelerations under the same circumstances. It is therefore useful to introduce concepts of "mass" and "force" *defined* by

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} \tag{1-1}$$

where m is a constant characteristic of each particle and called the mass of the particle (chosen as unity for some standard particle) and $\mathbf{F}$ is the force which is then *defined* by (1–1) (as applied to the standard particle or one whose mass has been determined by comparison with the standard under identical conditions).

Newton's Third Law* states that if a body A exerts a force, $\mathbf{F}$, on body B then, conversely, body B exerts an equal and opposite force, $-\mathbf{F}$, on body A. ("To *every* action, there is an equal and opposite reaction.")

Unlike the definition (1–1), this is a statement of *fact* which withstood 200 years experimental investigation (but is now known to have exceptions—at least with the simple definition, (1–1)).

The term "body" used above ultimately refers to mass-points but, by summation or integration, may also be interpreted as a physical system of any degree of complexity.

Note finally the implication that the forces can be regarded as "caused" by other agencies.† This, too, is a question of fact and has been well borne out by experience. In greater detail, experiment indicates that forces "from" different agencies, which act on the same particle, may be added vectorially:

$$\mathbf{F}_{\text{total on } A} = \sum_i \mathbf{F}_{\text{on } A \text{ due to } i}$$

From Newton's Third Law it follows that if a system of particles is "isolated" in the sense that all forces are produced within the system‡ ("no external forces act") and are independent of external conditions, then $0 = \sum \mathbf{F}_i = \frac{d}{dt}\left(\sum m_i \frac{d\mathbf{r}_i}{dt}\right)$ or if $\mathbf{v}_i = d\mathbf{r}_i/dt$ (velocity) then

$$\sum_i m_i \mathbf{v}_i = \mathbf{P}, \text{ a constant} \tag{1-2}$$

* The first two are contained in (1–1) and are really a definition of "force."

† Philosophically better: certain forces are present when and only when the associated agencies are present (correlation, not causation).

‡ More precisely, if the Lagrangian function of the system is independent of translations of the coordinates describing the system.

(The law of conservation of "*momentum*", **P**.) Correspondingly, if we define the momentum of a particle as $\mathbf{p}_i = m_i\mathbf{v}_i$ then $\sum_i \mathbf{p}_i =$ constant if no external forces act.

The "*work*" done between times t_1 and t_2 by the force **F** acting on a particle which undergoes displacements d**r** (whether "due to" the force or not) is defined by

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{1-3}$$

From (1–1)

$$W = \tfrac{1}{2}mv^2\Big|_{t_1}^{t_2} = T_2 - T_1 \tag{1-4}$$

where

$$T = \tfrac{1}{2}mv^2 \tag{1-5}$$

is known as the "*kinetic energy*" of the particle. The work done on a particle is always equal to the change in its kinetic energy, according to (1–4).

The change in momentum $\mathbf{p} = m\mathbf{v}$ of a particle is always equal to

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t = (m\mathbf{v})_2 - (m\mathbf{v})_1 \tag{1-6}$$

which is called the "*impulse*".

Moving Coordinates, Coriolis Forces

All of the above relations are valid in general (since, indeed, most of them are definitions) but in moving coordinates one must remember that the "basis vectors", $\mathbf{e}_i$, in $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ are also moving so that their time derivatives enter along with those of the "components", x_i. Since (1–1) involves a double time derivative, some of the extra terms will appear multiplied by the (local, apparent) velocity of the particle, $\mathrm{d}x_i/\mathrm{d}t$. These extra "forces", which have the property that they appear only when the particle is in motion (as seen from the moving coordinates), are known as "Coriolis forces".

Potential Fields, Conservative Forces

In many circumstances **F** can be expressed as the gradient of a simple scalar function:

$$\mathbf{F} = -\nabla\phi(\mathbf{r}, t) \tag{1-7}$$

(evaluated at the position of the particle). Here $\phi(\mathbf{r}, t)$ is known as the "potential energy". If ϕ is actually independent of t, then from (1–4), (1–7) and (1–1),

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + \phi) = 0; \qquad T + \phi = E, \text{ a constant} \tag{1-8}$$

Here E is called the total energy of the particle. Forces which keep E constant are called "conservative forces".

Conversely, if the total energy is conserved, the forces must be derivable from some ϕ via (1–7). [Thus, if $|\mathbf{v}|$ depends only on **r**—for any orbit—then ϕ *defined* by (1–8) satisfies (1–7).]

Example: (Newton's law of gravity). The gravitational force which each of two "particles" exerts on the other is experimentally found to be

$$m_1 m_2 G \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \text{force on \#2 ``due to'' \#1}$$

Here G is a universal constant of Nature. Both forces are simply obtained from (1–7) if we take

$$-\phi = \frac{m_1 m_2 G}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{1–9}$$

Orbit-Tracing in Potential Fields

Let $U = (\text{constant} - \phi)$ so that $\mathbf{F} = +\nabla U$ and adjust the constant so that, for the orbit desired, $\frac{1}{2}mv^2 = U$. Then if the independent variable, t, in (1–1) is replaced by the path-length, s, the result is

$$2U \frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{r}}{ds} \times \left[(\nabla U) \times \frac{d\mathbf{r}}{ds}\right] \tag{1–10}$$

That is,

$$\text{Curvature vector} = \text{Projection of} \left(\frac{\mathbf{F}}{mv^2}\right) \text{ perpendicular to the orbit} \tag{1–11}$$

This relation can be made the basis of a numerical or graphical method for tracing the path of a particle under the influence of conservative static forces.

Constraints

In some cases, a particle is required to satisfy such conditions as $g_\alpha(\mathbf{r}, t) = 0$. (EXAMPLE: motions of a particle over a surface.) Such cases are often most easily treated by substituting these conditions directly in the equations of motion (or by using the Lagrange equations; see later).

Occasionally, however, Lagrange's method of undetermined multipliers is more convenient: We have $\nabla g_\alpha = 0$; multiply these by unspecified functions, $\lambda_\alpha(\mathbf{r}, t)$ and add to (1–1),

$$m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + \sum_\alpha \lambda_\alpha \nabla g_\alpha \tag{1–12}$$

These equations may often conveniently be solved by algebraically eliminating the λ_α and picking out solutions of the resulting relations which satisfy $g_\alpha(\mathbf{r}, t) = 0$. [If the constraining forces, $\lambda_\alpha \nabla g_\alpha$, are desired, they may then be found by returning to (1–12).]

Systems of Mass Points

As already hinted, a system of many mass points, m_i, is described by a set of equations (1–1), one for each particle, along with a prescription, perhaps of the form, (1–7), for the forces. The total momentum and total kinetic energy of the system are defined as the sum of those of the individual particles:

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i \qquad (i = 1, \cdots, n;\ 3n \text{ equations}) \tag{1–13}$$

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i; \qquad T = \sum_i \tfrac{1}{2} m_i v_i^2 \tag{1–14}$$

and if $\mathbf{F}_i = -\nabla_i \phi$ (that is, if $(\mathbf{F}_i)_x = \frac{\partial}{\partial x_i} \phi(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n)$, etc.) then

$$T + \phi = E, \text{ a constant, the energy of the system}$$

Note that the concept of potential of *a* particle of the system need not have meaning.

Center of Mass

If we define

$$\mathbf{r}_c = \frac{1}{M}\sum_i m_i\mathbf{r}_i \qquad \left(M = \sum_i m_i\right) \tag{1–15}$$

as the position of the "center of mass" of the system, then $\mathbf{P} = M(\mathrm{d}\mathbf{r}_c/\mathrm{d}t)$ and from (1–13) and (1–14),

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \sum_i \mathbf{F}_i\,; \qquad \text{or:} \qquad \mathbf{F} = M\frac{\mathrm{d}^2\mathbf{r}_c}{\mathrm{d}t^2} \tag{1–16}$$

where $\mathbf{F}$ is the total *external* force acting on all the particles. (By Newton's third law, the overall sum of internal forces is zero.) Thus *the center of mass of a system moves as if all the mass and all external forces were concentrated there.* Thus macroscopic bodies, whether rigid or not, can be treated as point particles, if we wish to know only the motion of the center of mass.

Kinetic Energy

From (1–14) and (1–15) it follows immediately that

$$T = \tfrac{1}{2}M|\mathbf{v}_c|^2 + \sum_i \tfrac{1}{2}m_i|\mathbf{v}_i - \mathbf{v}_c|^2 \tag{1–17}$$

(where $\mathbf{v}_c = \mathrm{d}\mathbf{r}_c/\mathrm{d}t$). That is, the kinetic energy of the system can be expressed as that of a mass point $M = \sum_i m_i$ concentrated at the center of mass plus the kinetic energy due to motions of the particles *relative* to the center of mass.*

Angular Momentum; Torque

To indicate gross features of the additional motions of the system around its center of mass, define†

$$\mathbf{J} = \sum_i m_i\mathbf{r}_i \times \mathbf{v}_i \qquad \mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{F}_i \tag{1–18}$$

called respectively the total "angular momentum" and total "torque" (or "moment") on the system. From (1–13), it follows that

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \mathbf{L} \tag{1–19}$$

as a "gross" equation for the motion about the center of mass.

Equation (1–19) is the anti-symmetric part of the more general dyadic relation (for each particle)

$$\frac{\mathrm{d}}{\mathrm{d}t}m\mathbf{r}\mathbf{v} = \mathbf{r}\mathbf{F} + m\mathbf{v}\mathbf{v} \tag{1–20}$$

which is also an immediate consequence of (1–1). The symmetric part of this relation is

$$-\mathbf{r}\cdot\mathbf{F} = 2T - \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{p})$$

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* In view of (1–16), it might appear that the extra term in (1–17) violates the general principle (1–4). It does not, of course, and the reason is that, despite (1–16), the external forces do not actually act *on* $\mathbf{r}_c$ and this must be taken into account in computing the work done by them.

† Note that the order of the factors in (1–18) is purely a matter of *convention*; the corresponding sign-ambiguity indicates that **J** and **L** are "axial" vectors.

If the system is such that $\mathbf{r} \cdot \mathbf{p}$ is bounded, then taking time averages $\left[(1/t_0)\int_0^{t_0} \cdots dt;\ t_0 \to \infty\right]$ gives the "virial theorem"; summing over all the particles of the system:

$$2\langle T\rangle_{\text{Av}} = -\left\langle \sum_i \mathbf{r}_i \cdot \mathbf{F}_i \right\rangle_{\text{Av}} \tag{1-21}$$

If both the positions and the velocities are taken relative to the center of mass, one obtains the same equation. Thus if

$$\mathbf{J}_c = \sum_i m_i(\mathbf{r}_i - \mathbf{r}_c) \times (\mathbf{v}_i - \mathbf{v}_c) = \mathbf{J} - \mathbf{r}_c \times \mathbf{P} \tag{1-22}$$

and if

$$\mathbf{L}_c = \sum_i (\mathbf{r}_i - \mathbf{r}_c) \times \mathbf{F}_i = \mathbf{L} - \mathbf{r}_c \times \mathbf{F} \tag{1-23}$$

then from (1–16) and (1–19)

$$\frac{d\mathbf{J}_c}{dt} = \mathbf{L}_c \tag{1-24}$$

quite independent of the motion of $\mathbf{r}_c$.

In particular, *if no external forces act* and the mutual (equal and opposite) forces between two particles lie on a line joining them,* then $\mathbf{L}_c = 0$ so that $\mathbf{J}_c$ *is constant.*

Rigid Bodies

For a *rigid* system of mass points, equations (1–16) and (1–24) completely determine the entire motion:

First, define the following tensor ("moment of inertia tensor"—relative to the center of mass),

$$\mathfrak{I}_c = \sum_i m_i[|\mathbf{r}_i - \mathbf{r}_c|^2\,\mathbf{1} - (\mathbf{r}_i - \mathbf{r}_c)(\mathbf{r}_i - \mathbf{r}_c)] \tag{1-25}$$

(the last term is *not* a dot product) where $\mathbf{1}$ is the unit tensor and $\mathbf{r}_c$ is the position of the center of mass.

Note that, just as $\mathbf{r}_c$ is defined independently of the coordinate system, so the tensor $\mathfrak{I}_c$ is completely defined by (1–25) quite independently of whether the body is in motion or not. If it is moving, the *components* of $\mathfrak{I}_c$ in a fixed coordinate system may change (as do those of $\mathbf{r}_c$) but the tensor "changes" only in the same sense that $\mathbf{r}_c$ changes.

For a rigid body, it follows from (10–26) that

$$\mathbf{v}_i - \mathbf{v}_c = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_c) \tag{1-26}$$

where $\boldsymbol{\omega}$ (which can be time-varying) is known as the instantaneous angular velocity of the body. From (1–26), (1–25), (1–22) and (1–17) it then follows that

$$\mathbf{J}_c = \mathfrak{I}_c \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathfrak{I}_c \tag{1-27}$$

and that†

$$T = \tfrac{1}{2}M|\mathbf{v}_c|^2 + \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathfrak{I}_c \cdot \boldsymbol{\omega} \tag{1-28}$$

The main advantage of these relations appears upon transforming to a *coordinate system fixed in the body.* Then the components of $\mathfrak{I}_c$ become constants and, moreover,

* More generally: if the Lagrangian function of the system is independent of rotations of the coordinates describing the system.

† Note the identity, $|\boldsymbol{\omega} \times \mathbf{a}|^2 = (\boldsymbol{\omega} \times \mathbf{a}) \cdot (\boldsymbol{\omega} \times \mathbf{a}) = [(\boldsymbol{\omega} \times \mathbf{a}) \times \boldsymbol{\omega}] \cdot \mathbf{a} = -\mathbf{a} \cdot [\boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{a}) - \mathbf{a}(\boldsymbol{\omega} \cdot \boldsymbol{\omega})] = \omega^2 a^2 - (\boldsymbol{\omega} \cdot \mathbf{a})^2$.

because these components form a symmetric (and thus Hermitian) matrix, there is one set of (orthogonal) coordinates, x, y, z in which $\mathfrak{I}_c$ is diagonal:

$$(\mathfrak{I}_c)_{jk} = I_j\, \delta_{jk} \qquad \text{(constants)} \tag{1–29}$$

The three constants, I_j, are known as the "principal" moments of inertia. It is easily shown [by taking the origin at $\mathbf{r}_c$ and writing out explicitly the "xx" component of (1–25)] that these constants are given by

$$I_{xx} = \sum_i m_i(y_i^2 + z_i^2), \quad \text{etc.} \tag{1–30}$$

Thus the principal moments of inertia are necessarily positive.

From (10–27) it follows that, even though the coordinates are fixed in the rotating body, the equations of motion (1–24) take, with the help of (1–29) and (1–27), the form

$$\mathbf{L}_c = \frac{d\mathbf{J}_c}{dt} = \frac{d_r\mathbf{J}_c}{dt} + \boldsymbol{\omega} \times \mathbf{J}_c = \mathfrak{I}_c \cdot \frac{d_r\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times [\mathfrak{I}_c \cdot \boldsymbol{\omega}] \tag{1–31}$$

or, in components:

$$\left.\begin{aligned} L_x &= I_{xx}\frac{d\omega_x}{dt} + (I_{zz} - I_{yy})\omega_y\omega_z \\ L_y &= I_{yy}\frac{d\omega_y}{dt} + (I_{xx} - I_{zz})\omega_z\omega_x \\ L_z &= I_{zz}\frac{d\omega_z}{dt} + (I_{yy} - I_{xx})\omega_y\omega_z \end{aligned}\right\} \tag{1–32}$$

These equations completely determine the motion of the rigid body about its center of mass. Note that the simple form (1–32) is obtained *only* if *principal* axes *fixed* to the body are used. Note also that if $\mathbf{L} = 0$, $\boldsymbol{\omega}$ is still not constant unless two of its three components (in *this* system) vanish; that is, the only stable axes of rotation for a free body are the three principal axes.

Rigid Body with One Fixed Point

If a body is "anchored" at one point* but is otherwise free, it becomes useful to take that point as the coordinate origin, $\mathbf{r} = 0$, and *then* define a moment of inertia tensor:

$$\mathfrak{I} = \sum_i m_i[r_i^2\mathbf{1} - \mathbf{r}_i\mathbf{r}_i] \tag{1–33}$$

Unlike $\mathfrak{I}_c$, this tensor does depend on the position of the coordinate *origin* but, of course, is otherwise independent of the coordinates or motion of the body.

Since $\mathbf{v} \equiv 0$ at $\mathbf{r} = 0$, it follows from (10–26) that $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$ for some (time-varying) $\boldsymbol{\omega}$, the instantaneous angular velocity. Then, as above,

$$\mathbf{J} = \boldsymbol{\omega} \cdot \mathfrak{I} = \mathfrak{I} \cdot \boldsymbol{\omega} \tag{1–34}$$

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathfrak{I} \cdot \boldsymbol{\omega} \tag{1–35}$$

Note that (1–35) includes the term, $\frac{1}{2}Mv_c^2$ of (1–28) and that both (1–34) and (1–35) are special consequences of the fact that $\mathbf{v} \equiv 0$ when $\mathbf{r} = 0$.

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* The point need not be actually a part of the body; in some special cases (gyro top in a cage, for example) an imaginary extension of the body may be used.