Lecture Notes in Physics

Edited by J. Ehlers, Austin, K. Hepp, Zürich and
H. A. Weidenmüller, Heidelberg
Managing Editor: W. Beiglböck, Heidelberg

8

Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics

September 15–19, 1970
University of California, Berkeley

Edited by Maurice Holt

Editor's Preface

This volume of Lecture Notes in Physics is devoted to the Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics held at the University of California, Berkeley, September 15-19, 1970. A total of 65 papers were presented and they are published here in full, with a minimum of editorial changes. The Conference was divided into seven sessions with two devoted to new fundamental numerical techniques, two to viscous flow problems, two to high speed compressible flow and one to incompressible flow. Contributions from many countries were made, including important papers from the USA, the USSR, France, Germany, England, Holland, Canada and Australia.

The Conference was organized within a one year period, following the First Conference held in Novosibirsk, USSR, in August 1969. Valuable financial support was provided through a grant from the U.S. Office of Naval Research and the U.S. Air Force Office of Scientific Research. Many services of the Berkeley campus of the University were made available to us, including the use of the Physical Sciences Lecture Hall for all Conference sessions, and housing in the Halls of Residence. The Northrop Corporation kindly provided refreshments at the opening reception of the Conference.

I wish to thank the many students and colleagues who worked to make the Conference a success. Special mention should be made of Mrs. Arlene Martin, who did all the secretarial work, Mr. William F. Ballhaus, Jr., a graduate student, who supervised most of the arrangements, and Drs. Mark Wilkins and Robert L. Street who served on the program committee.

Finally I am indebted to Dr. W. Beiglböck, Editor of Lecture Notes in Physics, and to Dr. Klaus Peters of Springer-Verlag, for arranging the early publication of these Proceedings in this series.

January 29, 1971

Editor's Preface

This volume of Lecture Notes in Physics is devoted to the Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics held at the University of California, Berkeley, September 15-19, 1970. A total of 65 papers were presented and they are published here in full, with a minimum of editorial changes. The Conference was divided into seven sessions with two devoted to new fundamental numerical techniques, two to viscous flow problems, two to high speed compressible flow and one to incompressible flow. Contributions from many countries were made, including important papers from the USA, the USSR, France, Germany, England, Holland, Canada and Australia.

The Conference was organized within a one year period, following the First Conference held in Novosibirsk, USSR, in August 1969. Valuable financial support was provided through a grant from the U.S. Office of Naval Research and the U.S. Air Force Office of Scientific Research. Many services of the Berkeley campus of the University were made available to us, including the use of the Physical Sciences Lecture Hall for all Conference sessions, and housing in the Halls of Residence. The Northrop Corporation kindly provided refreshments at the opening reception of the Conference.

I wish to thank the many students and colleagues who worked to make the Conference a success. Special mention should be made of Mrs. Arlene Martin, who did all the secretarial work, Mr. William F. Ballhaus, Jr., a graduate student, who supervised most of the arrangements, and Drs. Mark Wilkins and Robert L. Street who served on the program committee.

Finally I am indebted to Dr. W. Beiglböck, Editor of Lecture Notes in Physics, and to Dr. Klaus Peters of Springer-Verlag, for arranging the early publication of these Proceedings in this series.

January 29, 1971

erence

Lecture Notes in Physics

Bisher erschienen / Already published

Vol. 1: J. C. Erdmann, Wärmeleitung in Kristallen, theoretische Grundlagen und fortgeschrittene experimentelle Methoden. 1969. DM 20,–/ $ 5.50

Vol. 2: K. Hepp, Théorie de la renormalisation. 1969. DM 18,–/ $ 5.00

Vol. 3: A. Martin, Scattering Theory: Unitarity, Analyticity and Crossing. 1969. DM 14,–/ $ 3.90

Vol. 4: G. Ludwig, Deutung des Begriffs physikalische Theorie und axiomatische Grundlegung der Hilbertraumstruktur der Quantenmechanik durch Hauptsätze des Messens. 1970. DM 28,–/ $ 7.70

Vol. 5: M. Schaaf, The Reduction of the Product of Two Irreducible Unitary Representations of the Proper Orthochronous Quantummechanical Poincaré Group. 1970. DM 14,– / $ 3.90

Vol. 6: Group Representations in Mathematics and Physics. Edited by V. Bargmann. 1970. DM 24,– / $ 6.60

Vol. 7: R. Balescu, J. L. Lebowitz, I. Prigogine, P. Résibois, Z. W. Salsburg, Lectures in Statistical Physics. 1971. DM 18,– / $ 5.00

Vol. 8: Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics. Edited by M. Holt. 1971. DM 28,– / $ 7.70

Selected Issues from
Lecture Notes in Mathematics

Vol. 7: Ph. Tondeur, Introduction to Lie Groups and Transformation Groups. VIII, 176 pages. 1965. DM 13,50

Vol. 8: G. Fichera, Linear Elliptic Differential Systems and Eigenvalue Problems. IV, 176 pages. 1965. DM 13,50

Vol. 17: C. Müller, Spherical Harmonics. IV, 46 pages. 1966. DM 5,–

Vol. 25: R. Narasimhan, Introduction to the Theory of Analytic Spaces. IV, 143 pages. 1966. DM 10,–

Vol. 33: G. I. Targonski, Seminar on Functional Operators and Equations. IV, 110 pages. 1967. DM 10,–

Vol. 35: N. P. Bhatia and G. P. Szegö, Dynamical Systems. Stability Theory and Applications. VI, 416 pages. 1967. DM 24,–

Vol. 40: J. Tits, Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen. VI, 53 Seiten. 1967. DM 6,80

Vol. 45: A. Wilansky, Topics in Functional Analysis. VI, 102 pages. 1967. DM 9,60

Vol. 52: D. J. Simms, Lie Groups and Quantum Mechanics. IV, 90 pages. 1968. DM 8,–

Vol. 55: D. Gromoll, W. Klingenberg und W. Meyer, Riemannsche Geometrie im Großen. VI, 287 Seiten. 1968. DM 20,–

Vol. 56: K. Floret und J. Wloka, Einführung in die Theorie der lokalkonvexen Räume. VIII, 194 Seiten. 1968. DM 16,–

Vol. 60: Seminar on Differential Equations and Dynamical Systems. Edited by G. S. Jones. VI, 106 pages. 1968. DM 9,60

Vol. 71: Séminaire Pierre Lelong (Analyse), Année 1967-1968. VI, 190 pages. 1968. DM 14,– / $ 3.90

Vol. 81: J.-P. Eckmann et M. Guenin, Méthodes Algébriques en Mécanique Statistique. VI, 131 pages. 1969. DM 12,–

Vol. 82: J. Wloka, Grundräume und verallgemeinerte Funktionen. VIII, 131 Seiten. 1969. DM 12,–

Vol. 91: N. N. Janenko, Die Zwischenschrittmethode zur Lösung mehrdimensionaler Probleme der mathematischen Physik. VIII, 194 Seiten. 1969. DM 16,80

Vol. 102: F. Stummel, Rand- und Eigenwertaufgaben in Sobolewschen Räumen. VIII, 386 Seiten. 1969. DM 20,–

Vol. 103: Lectures in Modern Analysis and Applications I. Edited by C. T. Taam. VII, 162 pages. 1969. DM 12,–

Vol. 104: G. H. Pimbley, Jr., Eigenfunction Branches of Nonlinear Operators and their Bifurcations. II, 128 pages. 1969. DM 10,–

Vol. 109: Conference on the Numerical Solution of Differential Equations. Edited by J. Ll. Morris. VI, 275 pages. 1969. DM 18,– / $ 5.00

Vol. 116: Séminaire Pierre Lelong (Analyse) Année 1969. IV, 195 pages. 1970. DM 14,– / $ 3.90

Vol. 126: P. Schapira, Théorie des Hyperfonctions. XI, 157 pages. 1970. DM 14,– / $ 3.90

Vol. 127: I. Stewart, Lie Algebras. IV, 97 pages. 1970. DM 10,– / $ 2.80

Vol. 128: M. Takesaki, Tomita's Theory of Modular Hilbert Algebras and its Applications. II, 123 pages. 1970. DM 10,– / $ 2,80

Vol. 140: Lectures in Modern Analysis and Applications II. Edited by C. T. Taam. VI, 119 pages. 1970. DM 10,– / $ 2.80

Vol. 144: Seminar on Differential Equations and Dynamical Systems, II. Edited by J. A. Yorke. VIII, 268 pages. 1970. DM 20,– / $ 5.50

Vol. 145: E. J. Dubuc, Kan Extensions in Enriched Category Theory. XVI, 173 pages. 1970. DM 16,– / $ 4.40

Vol. 155: J. Horváth, Several Complex Variables, Maryland 1970, I. V, 214 pages. 1970. DM 18,– / $ 5.00

Vol. 159: R. Ansorge und R. Hass, Konvergenz von Differenzverfahren für lineare und nicht lineare Anfangswertaufgaben. VIII, 145 Seiten. 1970. DM 14,– / $ 3.90

Contents

Session I
Fundamental Numerical Techniques
A.J. CHORIN, Chairman

Session II
Numerical Techniques and Applications
V.V. RUSANOV, Chairman

Session VI
Navier-Stokes Equations.
Fully Viscous Flows
K. STEWARTSON, Chairman

Session VII
Incompressible Flow Problems
K. ROESNER, Chairman

Session I

Fundamental Numerical Techniques

A.J. Chorin, Chairman

ON THE GROUP CLASSIFICATION OF DIFFERENCE SCHEMES FOR SYSTEMS OF EQUATIONS IN GAS DYNAMICS

N. N. Yanenko and Y. I. Shokin

Computing Center, Academy of Sciences, USSR
Siberian Branch, Novosibirsk

INTRODUCTION

The subject of this paper is the group classification of difference schemes approximating the equations of gas dynamics. It is known that the equations of gas dynamics are invariant with respect to a certain group of point transformations in the space of independent and dependent variables. This invariance follows from the invariance of the conservation laws, which are the basis for the equations of gas dynamics. Any given difference scheme is utilized in connection with some particular grid which in itself upsets the invariance of a computational algorithm. This lack of invariance can be demonstrated, for example, in the calculation of critical features of the flow (shock waves, contact surfaces, weak discontinuities), which move with various inclinations to the grid lines. The introduction of the difference scheme makes a group analysis difficult because difference operators possess group properties different from differential operators. Consequently, it appeared desirable to carry out a group classification of difference schemes on the basis of their first differential approximation. An explanation of the first differential approximation was given in References [1] - [3] and proved fruitful in examining properties of stability and especially dissipation of difference schemes. Inasmuch as the first differential approximation is in fact a differential equation with coefficients containing the parameters of the scheme, it occupies an intermediate position between the basic equations of gas dynamics and the difference scheme approximating them: In its hyperbolic part it preserves information concerning the basic equations while the difference scheme is reflected in the parabolic part. In consequence the obvious question is, to what extent does the first differential approximation preserve the group characteristics of the equations of gas dynamics. In this connection, all schemes can be divided into two classes: those which preserve the group characteristics and those which do not. In this paper we formulate those conditions under which a parabolic system of equations of the first differential approximation admits all groups of transformations [4], allowed by the basie system of gas dynamic equations. Systems of equations in one space variable are studied in both Eulerian and Lagrangian coordinates; in the case of two space dimensions only the Eulerian point of view is taken. Furthermore, it is noted in which systems of Lagrangian equations of the first differential approximation the law of conservation of mass is observed and where it is violated. Stability of the classes of schemes constructed is tested by the method of the first differential approximation.

1. THE CASE OF ONE SPACE DIMENSION

1. Let us consider the system of equations of gas dynamics in Eulerian coordinates in the case of one space variable.

$$w_t = f_x \quad , \tag{1.1}$$

in which

$$w = \begin{pmatrix} \rho u \\ \rho \\ \rho E \end{pmatrix}, \quad f = \begin{pmatrix} -p-\rho u^2 \\ -\rho u \\ -\rho u E - up \end{pmatrix}, \quad E = \varepsilon + \frac{1}{2}u^2,$$

u is the gas velocity, p is the pressure, ρ is density, ε is the specific internal energy. The equation of state of the gas has the form

$$\varepsilon = \varepsilon(p,\rho) \quad .$$

The system of equations (1) is approximated by the following difference scheme:

$$\frac{\Delta_o w^n(x)}{\tau} = \frac{\Delta_1 + \Delta_{-1}}{2h} f^n(x) + \frac{\Omega(x + \frac{h}{2})\,\frac{\Delta_1}{h} - \Omega(x - \frac{h}{2})\,\frac{\Delta_{-1}}{h}}{h} w^n(x) \quad , \tag{1.2}$$

in which $t = n\tau$, τ is the step in time t, h is the length step along the x axis, $w^n(x) = w(x, n\tau)$, $\Omega = ||\Omega_{ij}||_1^3$ is a matrix as yet unknown, for which $||\Omega|| = O(\tau)$,

$$\Delta_o = T_o - E \ , \quad \Delta_1 = T_1 - E \ , \quad \Delta_{-1} = E - T_{-1} \ ,$$

T_o is the displacement operator in t, T_1 is the displacement operator in x, E is the identity operator, $T_{-1} = T_1^{-1}$. The difference scheme (2) has at least first order accuracy.

The hyperbolic and parabolic forms of the first differential approximation of the difference scheme (1.2) have, respectively, the forms:

$$\frac{\tau}{2} w_{tt} + w_t = f_x + (\Omega w_x)_x \ , \tag{1.3}$$

$$w_t = f_x + (Cw_x)_x \ , \tag{1.4}$$

in which

$$C = \Omega - \frac{\tau}{2} A^2 = ||\mu_{ij}||_1^3 \ , \quad A = \frac{df}{dw} \ ,$$

$$\mu_{11} = \Omega_{11} - \frac{\tau}{2}(3u^2 + \theta + z\eta - 3u^2 z) \ ,$$

$$\mu_{12} = \Omega_{12} + \frac{\tau}{2} u[2u^2 - 2\theta - 3u^2 z + 3Ez + z\eta] \ ,$$

$$\mu_{13} = \Omega_{13} - \frac{\tau}{2} \tau u z \ , \quad \mu_{21} = \Omega_{21} - \frac{\tau}{2} u(2-z)$$

$$\mu_{22} = \Omega_{22} + \frac{\tau}{2}(u^2 - \theta - u^2 z + Ez) \ ,$$

$$\mu_{23} = \Omega_{23} - \frac{\tau}{2} z \ ,$$

$$\mu_{31} = \Omega_{31} - \tau u(E + \eta - u^2 z) - \frac{\tau}{2} u(\theta - Ez) \ ,$$

$$\mu_{32} = \Omega_{32} + \frac{\tau}{2}[(2u^2 + Ez)(E + \eta) - (u^2 + E)\theta - \eta\theta - u^2(2u^2 - E)z] \ ,$$

$$\mu_{33} = \Omega_{33} - \frac{\tau}{2} u^2(1 + 2z) - \frac{\tau}{2} z(E + \eta) \ ,$$

$$z = \frac{p_\varepsilon}{\rho} \ , \quad \eta = \frac{p}{\rho} \ , \quad \theta = p_\rho \ ,$$

$$\Omega_{ij} = \Omega_{ij}(t, x, w, w_t, w_x, \dots) \ .$$

The system of equations (1.4) can be written in the form

$$w_t = f_x + N_x \quad , \tag{1.5}$$

in which

$$N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \overline{C}\,\overline{w}_x = C\,w_x \quad , \quad \overline{w} = \begin{pmatrix} u \\ \rho \\ p \end{pmatrix} \quad , \quad \overline{C} = \|\delta_{ij}\|_1^3 ,$$

$$N_k = \delta_{k1}\,u_x + \delta_{k2}\,\rho_x + \delta_{k3}\,p_x \quad ,$$

$$\delta_{k1} = \rho(\mu_{k1} + u\,\mu_{k3}) \quad ,$$

$$\delta_{k2} = u\,\mu_{k1} + \mu_{k2} + (E + \rho\,\varepsilon_\rho)\mu_{k3} \quad ,$$

$$\delta_{k3} = \rho\,\varepsilon_p\,\mu_{k3} \quad .$$

Expression of the functions w,f in terms of the function $\overline{w}$ leads to the result that the system of equations (1.5) is equivalent to the system

$$F_1 = u_t + u\,u_x + \frac{1}{\rho}\,p_x - \frac{1}{\rho}\,N_{1x} + \frac{u}{\rho}\,N_{2x} = 0 \quad ,$$

$$F_2 = \rho_t + u\,\rho_x + \rho\,u_x - N_{2x} = 0 \quad , \tag{1.6}$$

$$F_3 = p_t + u\,p_x + a^2\,u_x + b\,N_{2x} + \frac{u}{\rho\,\varepsilon_\rho}\,N_{1x} - \frac{1}{\rho\,\varepsilon_p}\,N_{3x} = 0 \quad ,$$

in which

$$a^2 = \frac{p - \rho^2\,\varepsilon_\rho}{\rho\,\varepsilon_p} \quad , \qquad b = \frac{\varepsilon + \rho\,\varepsilon_\rho - \frac{1}{2}\,u^2}{\rho\,\varepsilon_p} \quad .$$

2. The system of equations (1.1) admits a space of operators with the basis [4]:

$$L_1 = \frac{\partial}{\partial t} \quad , \quad L_2 = \frac{\partial}{\partial x} \quad , \quad L_3 = t\,\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \quad , \quad L_4 = t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} \quad , \tag{1.7}$$

which represent, respectively, the following finite transformations preserving the system of equations (1.1):

1) translation in time,
2) translation in a space coordinate,
3) Galilean transformation,
4) Similarity transformation.

The requirement that the system of equations (1.4) or the equivalent system (1.6) admit a space of operators with the basis (1.7) leads to certain restrictions on the choice of matrix Ω.

The system of equations (1.6) admits a space of operators with the basis (1.7) if, and only if, the following equations are satisfied (see [4]):

$$\tilde{L}_\alpha\,F_k \Big|_{F_1=0,F_2=0,F_3=0} = 0 \quad ,$$

$$\alpha = 1, 2, 3, 4 \quad ; \quad k = 1, 2, 3 \quad ,$$

in which $\tilde{L}_\alpha$ is the extended operator obtained by extending the operator L_α.

In the given case

$$\tilde{L}_1 = L_1 = \frac{\partial}{\partial t} \quad , \quad \tilde{L}_2 = L_2 = \frac{\partial}{\partial x} \quad ,$$

$$\tilde{L}_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - u_x \frac{\partial}{\partial u_t} - \rho_x \frac{\partial}{\partial \rho_t} - p_x \frac{\partial}{\partial p_t} \quad ,$$

$$\tilde{L}_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u_t \frac{\partial}{\partial u_t} - \rho_t \frac{\partial}{\partial \rho_t} - p_t \frac{\partial}{\partial p_t} - u_x \frac{\partial}{\partial u_x} - \rho_x \frac{\partial}{\partial \rho_x} - p_x \frac{\partial}{\partial p_x} \quad .$$

Lemma. If in the difference scheme (1.2) the elements of the matrix Ω are independent of t and x, the system of equations (1.6) is invariant with respect to transformations consisting of translation in either the time or space coordinates.

Indeed, in satisfying the conditions of the lemma, it follows that

$$\tilde{L}_1 F_1 = -\frac{1}{\rho}\frac{\partial}{\partial t} N_{1x} + \frac{u}{\rho}\frac{\partial}{\partial t} N_{2x} = 0 \quad ,$$

$$\tilde{L}_2 F_1 = -\frac{1}{\rho}\frac{\partial}{\partial x} N_{1x} + \frac{u}{\rho}\frac{\partial}{\partial x} N_{2x} = 0 \quad ,$$

$$\tilde{L}_1 F_2 = -\frac{\partial}{\partial t} N_{2x} = 0 \quad , \quad \tilde{L}_2 F_2 = -\frac{\partial}{\partial x} N_{2x} = 0 \quad ,$$

$$L_1 F_3 = b\frac{\partial}{\partial t} N_{2x} + \frac{u}{\rho\, \varepsilon_p}\frac{\partial}{\partial t} N_{1x} - \frac{1}{\rho\, \varepsilon_p}\frac{\partial}{\partial t} N_{3x} = 0 \quad ,$$

$$\tilde{L}_2 F_3 = b\frac{\partial}{\partial x} N_{2x} + \frac{u}{\rho\, \varepsilon_p}\frac{\partial}{\partial x} N_{1x} - \frac{1}{\rho\, \varepsilon_p}\frac{\partial}{\partial x} N_{3x} = 0 \quad ,$$

inasmuch as the independence of the elements of the matrix Ω on x and t implies also independence of δ_{ij} on x,t, thus proving the lemma.

Theorem 1.1. If the conditions of the lemma are satisfied, and if in addition,

$$\frac{\partial}{\partial u_t}\Omega_{ij} = 0 \quad , \quad \frac{\partial}{\partial \rho_t}\Omega_{ij} = 0 \quad , \quad \frac{\partial}{\partial p_t}\Omega_{ij} = 0 \quad (i,j = 1,2,3) \quad , \qquad (1.8)$$

$$\frac{\partial}{\partial u} N_{1x} = N_{2x} \quad , \quad \frac{\partial}{\partial u} N_{2x} = 0 \quad , \quad \frac{\partial}{\partial u} N_{3x} = N_{1x} \quad , \qquad (1.9)$$

$$\left(u_x \frac{\partial}{\partial u_x} + \rho_x \frac{\partial}{\partial \rho_x} + p_x \frac{\partial}{\partial p_x}\right) N_{\alpha x} = N_{\alpha x} \quad (\alpha = 1,2,3) \quad , \qquad (1.10)$$

the system of equations (1.6) of the first differential approximation of the difference scheme (1.2) admits a space of operators with the basis (1.7).

Proof. Under our assumptions it follows from the lemma that

$$\tilde{L}_1 F_k = 0 \quad , \quad \tilde{L}_2 F_k = 0 \quad (k = 1,2,3) \quad .$$

A further examination of the operator $\tilde{L}_3$ shows that

$$\tilde{L}_3 F_1 = - u_x + u_x - \frac{1}{\rho}\tilde{L}_3 N_{1x} + \frac{u}{\rho}\tilde{L}_3 N_{2x} + \frac{1}{\rho} N_{2x} = - \frac{1}{\rho}(\tilde{L}_3 N_{1x} - N_{2x}) + $$

$$+ \frac{u}{\rho}\tilde{L}_3 N_{2x} = - \frac{1}{\rho}\left(\frac{\partial}{\partial u} N_{1x} - N_{2x}\right) + \frac{u}{\rho}\frac{\partial}{\partial u} N_{2x} = 0 ,$$

$$\tilde{L}_3 F_2 = - \rho_x + \rho_x - \tilde{L}_3 N_{2x} = - \frac{\partial}{\partial u} N_{2x} = 0 ,$$

$$\tilde{L}_3 F_3 = - p_x + p_x + b\,\tilde{L}_3 N_{2x} - \frac{u}{\rho\varepsilon_p} N_{2x} + \frac{u}{\rho\varepsilon_p}\tilde{L}_3 N_{1x} + \frac{1}{\rho\varepsilon_p} N_{1x} -$$

$$- \frac{1}{\rho\varepsilon_p}\tilde{L}_3 N_{3x} = \frac{u}{\rho\varepsilon_p}\left(\frac{\partial}{\partial u} N_{1x} - N_{2x}\right) + \frac{1}{\rho\varepsilon_p}\left(N_{1x} - \frac{\partial}{\partial u} N_{3x}\right) = 0$$

Let

$$L_4 = u_x \frac{\partial}{\partial u_x} + \rho_x \frac{\partial}{\partial \rho_x} + p_x \frac{\partial}{\partial p_x} ,$$

Then

$$\tilde{L}_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u_t\frac{\partial}{\partial u_t} - \rho_t\frac{\partial}{\partial \rho_t} - p_t\frac{\partial}{\partial p_t} - L_4 ,$$

$$\tilde{L}_4 F_2 = - \rho_t - u\rho_x - \rho u_x - \tilde{L}_4 N_{2x} = - N_{2x} + L_4 N_{2x} = 0 ,$$

$$\tilde{L}_4 F_1 = - u_t - u u_x - \frac{1}{\rho} p_x - \frac{1}{\rho}\tilde{L}_4 N_{1x} + \frac{u}{\rho}\tilde{L}_4 N_{2x} =$$

$$= - \frac{1}{\rho} N_{1x} + \frac{u}{\rho} N_{2x} + \frac{1}{\rho} L_4 N_{1x} - \frac{u}{\rho} L_4 N_{2x} = 0 ,$$

$$\tilde{L}_4 F_3 = b(n_{2x} - L_4 N_{2x}) - \frac{u}{\rho\varepsilon_p}(L_4 N_{1x} - N_{1x}) + \frac{1}{\rho\varepsilon_p}(L_4 N_{3x} - N_{3x}) = 0$$

and the theorem is proved.

The conditions $(\partial/\partial u) N_{2x} = 0$ imply that N_{2x} does not depend explicitly on the function u. Then Eqs. (1.9) can be written in the form:

$$\frac{\partial}{\partial u} N_{2z} = 0 , \quad N_{1x} = u N_{2x} + R_1 , \quad N_{3x} = \frac{1}{2} u^2 N_{2x} + u R_1 + R_2 ,$$

$$\frac{\partial}{\partial u} R_1 = 0 , \quad \frac{\partial}{\partial u} R_2 = 0 . \qquad (1.11)$$

If

$$\Omega_{21} = \overline{\Omega}_{21} + \frac{\tau}{2} u(2-z) - u\left(\overline{\Omega}_{23} - \frac{\tau}{2} z\right) ,$$

$$\Omega_{22} = \overline{\Omega}_{22} - u\left[\overline{\Omega}_{21} - u\left(\overline{\Omega}_{23} - \frac{\tau}{2} z\right)\right] - \frac{\tau}{4} u^2 (2-z) - \frac{1}{2} u^2\left(\overline{\Omega}_{23} - \frac{\tau}{2} z\right) , \qquad (1.12)$$

$$\Omega_{23} = \overline{\Omega}_{23} ,$$

in which

$$\frac{\partial}{\partial u}\overline{\Omega}_{2k} = 0 \qquad (k = 1,2,3) , \qquad (1.13)$$

then

$$\frac{\partial}{\partial u} N_{2x} = 0 .$$

Thus Theorem 1.1 is still valid if, in its formulation, conditions (1.9) are replaced by the conditions (1.11) to (1.13).

3. Let it be required that, in the first differential approximation of the difference scheme (1.2), the law of conservation of mass is satisfied, that is, that equation

$$N_2 = 0$$

holds, and consequently,

$$\begin{aligned} \Omega_{21} &= \frac{\tau}{2} u(2-z) , \\ \Omega_{22} &= \frac{\tau}{2} (\theta - \varepsilon z - u^2 + u^2 z) , \\ \Omega_{23} &= \frac{\tau}{2} z . \end{aligned} \tag{1.14}$$

Then the system of equations (1.6) assumes the form:

$$\begin{aligned} F_1 &= u_t + uu_x + \frac{1}{\rho} p_x - \frac{1}{\rho} N_{1x} = 0 , \\ F_2 &= \rho_t + u\rho_x + \rho u_x = 0 , \\ F_3 &= p_t + up_x + a^2 u_x + \frac{u}{\rho \varepsilon_p} N_{1x} - \frac{1}{\rho \varepsilon_p} N_{3x} = 0 \end{aligned} \tag{1.15}$$

Theorem 1.2. If the conditions of the lemma, Eqs. (1.8), (1.10), (1.14), are satisfied, and if in addition,

$$\frac{\partial}{\partial u} N_{1x} = 0 , \quad N_{3x} = (u N_1)_x + R - u_x N_1 , \quad \frac{\partial}{\partial u} R = 0 , \tag{1.16}$$

then the system of equations (1.15) of the first differential approximation to difference scheme (1.2) admits a space of operators with the basis (1.7) and, in this approximation, the law of conservation of mass is satisfied.

Indeed, it follows from (1.14) that

$$\delta_{2k} = 0 \qquad (k = 1,2,3) ,$$

That is, $N_2 = 0$, and hence, in the first differential approximation the law of conservation of mass is satisfied. In addition the validity of Theorem 1.2 follows on the basis of Theorem 1.1, since in this case the condition (1.9) assumes the form

$$\frac{\partial}{\partial u} N_{1x} = 0 , \quad \frac{\partial}{\partial u} N_{3x} = N_{1x} ,$$

that is,

$$N_{3x} = u N_{1x} + R = (u N_1)_x + R - u_x N_1 , \quad \frac{\partial}{\partial u} R = 0 . \tag{1.17}$$

The fact that N_{1x} is independent of the function u implies that its coefficients δ_{ik} (k = 1,2,3) do not depend on the function u. Setting