THE TEACHING OF MATHEMATICS

A SOURCE BOOK AND GUIDE

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BY RALEIGH SCHORLING

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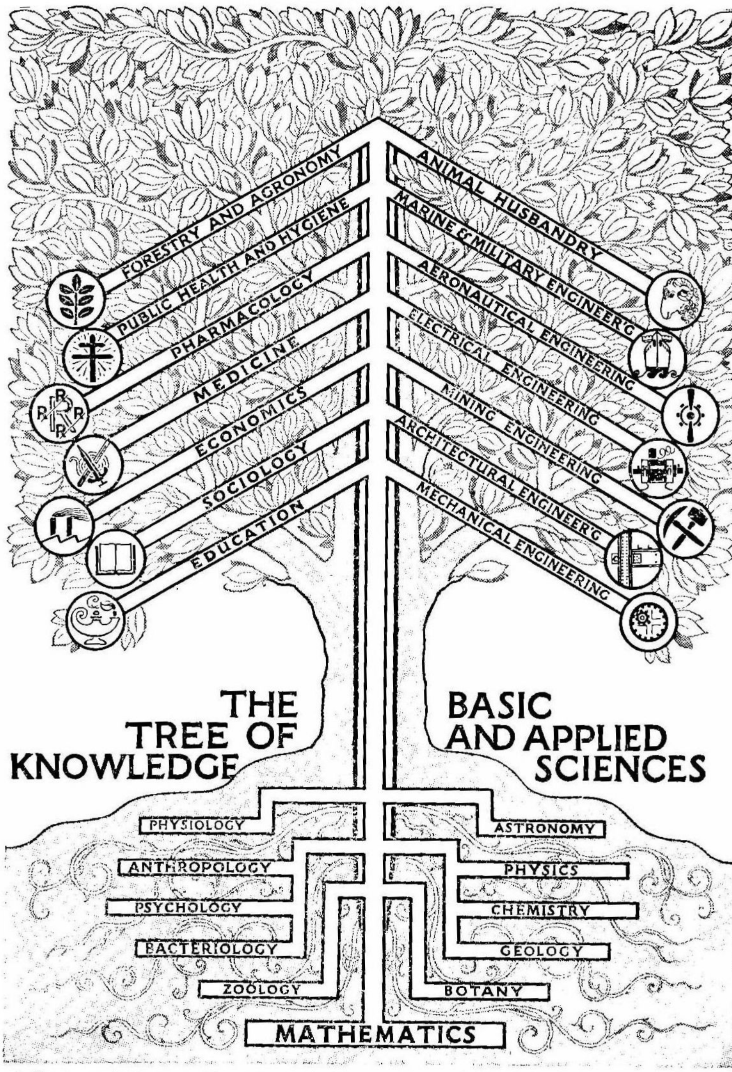

A Century of Progress Mural by John Norton, 1933 ©Museum of Science and Industry

THE TREE OF KNOWLEDGE

PREFACE

The need for a source book on the teaching of mathematics has been obvious for a long time. There are many fine things in pedagogy that need to be made more easily available for all teachers of mathematics in junior and senior high schools. This book, of course, does not include all the things relating to the teaching of mathematics that are fine and helpful. It does, however, include some of the things that have been widely read and greatly appreciated by teachers of mathematics. In Part I there is a reprint of a statement on Mathematical Requirements. This statement of aims and purposes is one of the best that can be found for any of the high school subjects. As a member of this committee the writer knows of the wearying hours that the gifted minds of Young, Smith, Tyler, and others gave to the careful scrutinizing of each word and phrase. The next chapter is a reprint of Professor David Eugene Smith's charming essay on "Mathematics in the Training for Citizenship." The fact that this material has been reprinted at least three times would seem to be clear evidence of its importance.

In Part II there appears a reprint of nearly all of Professor Moore's presidential addresses in which he laid so carefully the foundation for the mathematical reform that has challenged the best efforts of progressive groups in many centers for a quarter of a century. These are samples of materials that every teacher of mathematics should know.

But a source book built by reprinting articles in the literature would leave many gaps. The present volume attempts to fill these gaps by lists of guides and suggestions for techniques that have been drawn from the psychology of mathematics, from educational research, from committee reports, the journals, and the various yearbooks. Parts III and IV are devoted to the everyday problems of the classroom teacher of mathematics. The effort has been to put into a single volume the finest of the old and the most useful of the new.

It has been no easy task to translate the findings of educational research into classroom practice. Such findings have in the main been presented in the form of summaries employing a vocabulary that is technical and meaningless to the classroom teacher. The teacher is beginning to ask the research worker whether he really has anything to contribute to the improvement of instruction. The research worker is quite sure he has a good deal that is valuable, but he admits that there is a wide gap between practice and theory caused by his inability to state the outcomes of his research in an understandable way. The typical classroom teacher cannot and will not read highly technical material, nor should he be criticized for not doing so; for there has been no effort to present the more significant findings from the point of view of the classroom teacher, and there have not been enough illustrations and applications to make these findings meaningful. In the last analysis the screening of the wheat from the chaff in educational research is a matter of selection and opinion. There are vast areas in which research has little to offer, and there are many issues on which the findings are not clear. This volume, however, represents a first attempt to apply what is known, as an outcome of research or competent opinion, to practical classroom situations.

I am deeply indebted to Miss Martha Hildebrandt and Professor William David Reeve who, on behalf of the National Council of Teachers of Mathematics, gave me permission to reprint materials from the various yearbooks and from the *Mathematics Teacher.* I am obligated to Professor Walter O. Shriner for the summary style of writing employed in certain sections of Parts III and IV. I also take pleasure in acknowledging the courtesy of Mr. William Betz, Professor Ernst Breslich, Dr. John R. Clark, Professor Louis C. Karpinski, Professor David Eugene Smith, Professor Edward Lee Thorndike, the University of Chicago Press, Ginn and Company, the Macmillan Company, and the World Book Company, in permitting various quotations.

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PART I. AIMS, PRINCIPLES, AND GENERAL OBJECTIVES

CHAPTER I

THE AIMS OF MATHEMATICAL INSTRUCTION[1]

It has been customary to distinguish three classes of aims: (1) practical or utilitarian, (2) disciplinary, (3) cultural; and such a classification is indeed a convenient one. It should be kept clearly in mind, however, that the three classes mentioned are not mutually exclusive and that convenience of discussion rather than logical necessity often assigns a given aim to one or the other of the classes. Indeed any truly disciplinary aim is practical, and in a broad sense the same is true of cultural aims.

Practical Aims.—By a practical or utilitarian aim, in the narrower sense, we mean then the immediate or direct usefulness in life of a fact, method or process in mathematics.

1. The immediate and undisputed utility of the fundamental processes of arithmetic in the life of every individual demands our first attention. The first instruction in these processes, it is true, falls outside the period of instruction which we are considering. By the end of the sixth grade the child should be able to carry out the four fundamental operations with integers and with common and decimal fractions accurately and with a fair degree of speed. This goal can be reached in all schools—as it is being reached in many—if the work is done under properly qualified teachers and if drill is confined to the simpler cases which alone are of importance in the practical life of the great majority. Accuracy and facility in numerical computation are of such vital importance, however, to every individual that effective drill in this subject should be continued throughout the secondary school period, not in general as a separate topic, but in connection with the numerical problems arising in other work. In this numerical work, besides accuracy and speed, the following aims are of the greatest importance:

(*a*) A progressive increase in the pupil's understanding of the nature of the fundamental operations and power to apply them in new situa-

[1] This chapter is reprinted from *The Reorganization of Mathematics in Secondary Education,* A Report by the National Committee on Mathematical Requirements, pp. 6–12.

tions. The fundamental laws of algebra are a potent influence in this direction. (See 3, below.)

(*b*) Exercise of common sense and judgment in computing from approximate data, familiarity with the effect of small errors in measurements, the determination of the number of figures to be used in computing and to be retained in the result, and the like.

(*c*) The development of self-reliance in the handling of numerical problems, through the consistent use of checks on all numerical work.

2. Of almost equal importance to every educated person is an understanding of the language of algebra and the ability to use this language intelligently and readily in the expression of such simple quantitative relations as occur in everyday life and in the normal reading of the educated person.

Appreciation of the significance of formulas and ability to work out simple problems by setting up and solving the necessary equations must nowadays be included among the minimum requirements of any program of universal education.

3. The development of the ability to understand and to use such elementary algebraic methods involves a study of the fundamental laws of algebra and at least a certain minimum of drill in algebraic technique, which, when properly taught, will furnish the foundation for an understanding of the significance of the processes of arithmetic already referred to. The essence of algebra as distinguished from arithmetic lies in the fact that algebra concerns itself with the operations upon numbers *in general,* while arithmetic confines itself to operations on *particular* numbers.

4. The ability to understand and interpret correctly graphical representations of various kinds, such as nowadays abound in popular discussions of current scientific, social, industrial, and political problems will also be recognized as one of the necessary aims in the education of every individual. This applies to the representation of statistical data, which is becoming increasingly important in the consideration of our daily problems, as well as to the representation and understanding of various sorts of dependence of one variable quantity upon another.

5. Finally, among the practical aims to be served by the study of mathematics should be listed familiarity with the geometric forms common in nature, industry, and life; the elementary properties and relations of these forms, including their mensuration; the development of space-perception; and the exercise of spatial imagination. This involves acquaintance with such fundamental ideas as congruence and

similarity and with such fundamental facts as those concerning the sum of the angles of a triangle, the Pythagorean proposition and the areas and volumes of the common geometric forms.

Among directly practical aims should also be included the acquisition of the ideas and concepts in terms of which the quantitative thinking of the world is done, and of ability to think clearly in terms of those concepts. It seems more convenient, however, to discuss this aim in connection with the disciplinary aims.

Disciplinary Aims.—We would include here those aims which relate to mental training, as distinguished from the acquisition of certain specific skills discussed in the preceding section. Such training involves the development of certain more or less general characteristics and the formation of certain mental habits which, besides being directly applicable in the setting in which they are developed or formed, are expected to operate also in more or less closely related fields—that is, to "transfer" to other situations.

The subject of the transfer of training has for a number of years been a very controversial one. Only recently has there been any evidence of agreement among the body of educational psychologists. . . . It is sufficient for our present purpose to call attention to the fact that most psychologists have abandoned two extreme positions as to transfer of training. The first asserted that a pupil trained to reason well in geometry would thereby be trained to reason equally well in any other subject; the second denied the possibility of any transfer, and hence the possibility of any general mental training. That the effects of training do transfer from one field of learning to another is now, however, recognized. The amount of transfer in any given case depends upon a number of conditions. If these conditions are favorable, there may be considerable transfer, but in any case the amount of transfer is difficult to measure. Training in connection with certain attitudes, ideals, and ideas is almost universally admitted by psychologists to have general value. It may, therefore, be said that, with proper restrictions, general mental discipline is a valid aim in education.

The aims which we are discussing are so important in the restricted domain of quantitative and spatial (i.e., mathematical or partly mathematical) thinking which every educated individual is called upon to perform that we do not need for the sake of our argument to raise the question as to the extent of transfer to less mathematical situations.

In formulating the disciplinary aims of the study of mathematics the following should be mentioned:

(1) The acquisition, in precise form, of those ideas or concepts in terms of which the quantitative thinking of the world is done. Among these ideas and concepts may be mentioned ratio and measurement (lengths, areas, volumes, weights, velocities, and rates in general, etc.), proportionality and similarity, positive and negative numbers, and the dependence of one quantity upon another.

(2) The development of ability to think clearly in terms of such ideas and concepts. This ability involves training in—

(*a*) Analysis of a complex situation into simpler parts. This includes the recognition of essential factors and the rejection of the irrelevant.

(*b*) The recognition of logical relations between interdependent factors and the understanding and, if possible, the expression of such relations in precise form.

(*c*) Generalization; that is, the discovery, and formulation of a general law and an understanding of its properties and applications.

(3) The acquisition of mental habits and attitudes which will make the above training effective in the life of the individual. Among such habitual reactions are the following: A seeking for relations and their precise expression; an attitude of enquiry; a desire to understand, to get to the bottom of a situation; concentration and persistence; a love for precision, accuracy, thoroughness, and clearness, and a distaste for vagueness and incompleteness; a desire for orderly and logical organization as an aid to understanding and memory.

(4) Many, if not all, of these disciplinary aims are included in the broad sense of the idea of relationship or dependence—in what the mathematician in his technical vocabulary refers to as a "function" of one or more variables. Training in "functional thinking," that is, thinking in terms of relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics.

Cultural Aims.—By cultural aims we mean those somewhat less tangible but none the less real and important intellectual, ethical, esthetic or spiritual aims that are involved in the development of appreciation and insight and the formation of ideals of perfection. As will be at once apparent the realization of some of these aims must await the later stages of instruction, but some of them may and should operate at the very beginning.

More specifically we may mention the development or acquisition of—

(1) Appreciation of beauty in the geometrical forms of nature, art, and industry.

(2) Ideals of perfection as to logical structure; precision of state-

ment and of thought; logical reasoning (as exemplified in the geometric demonstration); discrimination between the true and the false, etc.

(3) Appreciation of the power of mathematics—of what Byron expressively called "the power of thought, the magic of the mind"[1]—and the rôle that mathematics and abstract thinking, in general, has played in the development of civilization; in particular in science, in industry, and in philosophy. In this connection mention should be made of the religious effect, in the broad sense, which the study of the permanence of laws in mathematics and of the infinite tends to establish.[2]

THE POINT OF VIEW GOVERNING INSTRUCTION

The practical aims enumerated above, in spite of their vital importance, may without danger be given a secondary position in seeking to formulate the general point of view which should govern the teacher, provided only that they receive due recognition in the selection of material and that the necessary minimum of technical drill is insisted upon.

The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

All topics, processes, and drill in technique which do not directly contribute to the development of the powers mentioned should be eliminated from the curriculum. It is recognized that in the earlier periods of instruction the strictly logical organization of subject matter[3] is of less importance than the acquisition, on the part of the pupil, of experience as to facts and methods of attack on significant problems, of the power to see relations, and of training in accurate thinking in terms of such relations. Care must be taken, however, through the dominance of the course by certain general ideas that it does not become a collection of isolated and unrelated details.

Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes, and

[1] D. E. Smith; "Mathematics in the Training for Citizenship," *Teachers College Record*, XVIII (May, 1917), 6.

[2] For an elaboration of the ideas here presented in the barest outline, the reader is referred to the article by D. E. Smith already mentioned and to his presidential address before the Mathematical Association of America, Wellesley, Mass., September 1, 1921.

[3] "The logical from the standpoint of subject matter represents the goal, the last term of training, not the point of departure." Dewey, *How We Think*, p. 62.

principles in the solution of concrete problems rather than on the acquisition of mere facility or skill in manipulation. The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. On the side of algebra, the ability to understand its language and to use it intelligently, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims. *Drill in alegbraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.* It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously.

On the side of geometry the formal demonstrative work should be preceded by a reasonable amount of informal work of an intuitive, experimental, and constructive character. Such work is of great value in itself; it is needed also to provide the necessary familiarity with geometric ideas, forms, and relations, on the basis of which alone intelligent appreciation of formal demonstrative work is possible.

The one great idea which is best adapted to unify the course is that of the *functional relation.* The concept of a variable and of the dependence of one variable upon another is of fundamental importance to everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data and the study of the formula and of the graph and of their uses.

The primary and underlying principle of the course should be the idea of relationship between variables, including the methods of determining and expressing such relationship. The teacher should have this idea constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends.

EXERCISES[1]

1. What is the customary classification of the general aims for teaching mathematics? How useful is this classification to the classroom teacher? To what extent are these aims claimed by teachers of science, American history, etc., for their subjects?

2. Illustrate the fact that these aims are not mutually exclusive.

3. Read the chapter on aims in a book on the teaching of mathematics printed twenty or more years ago. What changes in emphasis if any do you note? To what do some authors refer when they speak of "abiding values of mathematics"?

4. What are some of the reasons why the cultural values of mathematics are so seldom stressed in high school classes?

5. Assume that a school classifies its ninth grade pupils in three homogeneous groups—excellent, average, and dull. Would you emphasize cultural values or practical values if you were teaching mathematics to a class of dull pupils? Why?

6. Select five or six topics in a course in mathematics stressing cultural aims that dull pupils might study with profit and satisfaction.

7. What new ideas are suggested by Professor D. E. Smith's "Mathematics and Religion" in the *Sixth Yearbook* of the National Council of Teachers of Mathematics? Are there any statements that you think are invalid or "far-fetched"?

8. List five subsidiary aims that can be classified as practical.

9. How many main disciplinary aims are listed? Which one of these do you think is most important for high school pupils?

[1] The exercises in this book are suggestions for group discussions and for investigations to be carried out by individual students in preparation for conferences. The list is illustrative rather than comprehensive, for each instructor will wish to add such exercises to each list as his experiences and special interests may dictate.

Chapter II

MATHEMATICS IN THE TRAINING FOR CITIZENSHIP[1]

Purpose.—It is the purpose of this paper to set forth certain facts to be kept in mind in the teaching of mathematics for citizenship. It is proposed to consider in the limited space available what there is in mathematics that the working man needs and what of this science the woman requires in directing the education of her children and in managing her home; how mathematics trains the mind and how its poetry affects human life; what potency the subject has in the uplift of your soul and mine; how it has linked itself with all humanity in all times; and how we should go about to make all this real in our everyday teaching.

Thus stated, the problem falls of its own weight—for in the brief space allowed, it is impossible to treat with any adequacy these various topics and the many questions that naturally arise in your mind and my own. All that can be done is to mention a few of the leading principles that have guided the writer in his own endeavors to improve the teaching of mathematics, and which he feels sure guide a large proportion of his former students in their work in various parts of the world.

First Objective.—First, we study mathematics because it is one of the small group of subjects—like reading, history, and geography—that are linked up with a large number of the branches of human knowledge. No one can be happy as a member of the human family who does not know something of the history of the race, something of the earth on which this race exists, something of letters, something of the arts, and something of what we so pedantically call "the quantitative side of human life." Of the necessity for knowing number relations there can be no question, but fifty years ago one might well have cried the slogan abroad from the housetops, "Will anyone tell me why a girl should study algebra?" Today, a person would sadly feel his ignorance, or her ignorance, if he or she had to look with lack-luster eyes upon a simple formula such as may be found in *Popular Mechanics, Motor,* the *Scientific American,* an everyday article on the radio or astronomy, a boy's manual on the airplane, or any one of hundreds of articles in our popular encyclopedias. These needs come not only within the purview of the boy; they

[1] Reprinted here by permission of the author, Professor David Eugene Smith, from the *Third Yearbook* (1928) of the National Council of Teachers of Mathematics.