HANDBOOK OF MATHEMATICAL STATISTICS

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PREFACE

The study of a problem by statistical methods usually involves three stages: (1) the collection of material or data; (2) the mathematical analysis of the data thus collected; (3) the interpretation of results, for the particular purpose in view.

As to stage (1), the best methods of collecting data depend almost entirely on the nature of the particular field of inquiry, and are not discussed in this Handbook. The same is true in regard to stage (3); the problems connected with the interpretation of statistical results are necessarily very different in different fields of inquiry, and are not discussed in this Handbook, except as illustrations of the mathematical methods involved.

The problems of stage (2), on the other hand, are in a sense common to all fields of statistical inquiry. Whatever the content of the data may be, the *form* of the mathematical analysis is essentially the same. It is with these formal problems of mathematical analysis that this Handbook deals. Illustrations are taken from this or that particular field, for the sake of concreteness; but the *general* applicability of the methods to all fields is constantly borne in mind, and the terminology throughout the Handbook is kept as non-special as possible.

Special emphasis is laid on the limitations surrounding the proper application of the various methods of analysis. Without careful attention to these limitations, the results of a statistical inquiry may be altogether misleading.

Each chapter has been critically read by at least two other contributors besides the author; but the final responsibility for all the chapters rests with the individual authors.

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HANDBOOK OF MATHEMATICAL STATISTICS

CHAPTER I

MATHEMATICAL MEMORANDA

By E. V. HUNTINGTON

NUMERICAL COMPUTATION

Slide rules, tables, and computing machines. Before undertaking any statistical work one should supply one's self with suitable aids to computation.

For three-figure accuracy, a ten-inch slide rule is very convenient. The larger Fuller or Thacher slide rules give four, or sometimes five, significant figures. Barlow's Tables of squares, square roots, cube roots, and reciprocals, are almost indispensable. The multiplication tables are also often convenient. Crelle's Table gives the product of every three-figure number by every three-figure number. Peters's Table gives the product of every four-figure number by every two-figure number. The smaller table of H. Zimmermann gives the product of every three-figure number by every two-figure number. Tables of logarithms of numbers, and for certain purposes tables of trigonometric functions, are invaluable. Four- and five-place tables exist in great variety. If more than five figures are required, use Bremiker's six-place table or proceed at once to a seven-place table: for example, Vega. For eight places use the two-volume table of Bauschinger and Peters. Explanations of the use of tables of logarithms usually accompany the tables themselves; see, for example, E. V. Huntington's *Handbook of Mathematics for Engineers*.

In extended work some form of computing machine will soon pay for itself in spite of the apparently large initial expense. The best-known adding and listing machines are the Burroughs and the Wales, with standard keyboards, and the Dalton and the Sundstrand with ten-key keyboards. (The wide-paper form of carriage is more convenient for most purposes than the narrow-ribbon type.) Among the calculating machines may be mentioned the Comptometer, the Burroughs nonlisting machine, the Monroe calculator, the Millionaire, the Brunsviga, the Ensign, the Mercedes-Euclid, and the Marchant. Some of these can be operated by electricity.

For elaborate classification of large amounts of statistical data, as in the work of the Census Bureau, the Hollerith or the Powers machine for sorting punched cards is practically indispensable.

In advanced work in statistical theory, Pearson's *Tables for Statisticians and Biometricians* are invaluable.

The new *Tables for Applied Mathematics*, by J. W. Glover, include in one volume a large number of tables for finance, insurance, and statistics, together with a seven-place table of logarithms.

Absolute and relative errors. The numerical data in a statistical computation are usually the result of measurement, observation, or estimate, and hence are only approximately correct. The closeness of the approximation may be measured either by the absolute error or by the relative error.

The **absolute error** is sometimes defined as the observed value minus the true value $(x_1 - X)$ and sometimes as the true value minus the observed value $(X - x_1)$. When the distinction of sign is important, the error $x_1 - X$ may be called the **deviation** of the observed value from the true value (a positive deviation being an "error in excess," and a negative deviation an "error in defect"), while the error $X - x_1$ may be called the **correction** to be applied to the observed quantity (the correction being positive or negative according as the observed quantity needs to be increased or decreased).

The **relative error** is the absolute error divided by the true value.

For example, suppose $x_1 = 3.06$ cm. and $x_2 = 2.97$ cm. are two approximate values and $X = 3.00$ cm. is the true value. Then the absolute error of x_1 is 0.06 cm. (deviation $= + 0.06$ cm., correction $= - 0.06$ cm.) while the relative error is 0.02, or 2 per cent. Similarly, the absolute error of x_2 is 0.03 cm. (deviation $= - 0.03$ cm., correction $= + 0.03$ cm.) while the relative error is 0.01, or 1 per cent.

The absolute error is connected with the **number of decimal places,** and is important when the quantity is to be added or subtracted, or compared with other quantities on an absolute basis. For example, a measurement may be "correct to two decimal places"; an estimated population may be "correct to the nearest million," etc.

The relative error, on the other hand, is connected with the **number of significant figures,** and is important when the quantity is to be multiplied or divided, or compared with another quantity on a percentage basis. For example, a number may be said to be "correct to four significant figures," "correct to within 3 per cent of the value," "correct within one part in 6000," etc.

In any statistical investigation, either the desired number of decimal places, or more usually, the desired number of significant figures should be decided upon in advance, and borne constantly in mind throughout the work.

Propagation of error in computation. The manner in which small errors in the data may accumulate in the course of a computation is indicated as follows:

(1) In addition: Suppose, for example, that each of 20 numbers has a possible error of half a unit in the third decimal place; then the sum of these numbers may have a possible error of 10 units in the third decimal place — that is, an error of 1 unit in the second decimal place. All figures beyond the second decimal place should therefore be discarded in the answer. In general, one doubtful figure in any column will render that whole column doubtful; hence all figures to the right of that column should be discarded in the answer.

(2) In subtraction: Two numbers may each be correct, say, to five significant figures, and yet their difference may be correct to only one or two significant figures; for example, $3.1416 - 3.1402 = 0.0014$. Neglect of this fact is a frequent source of overconfidence in regard to the precision of a result.

(3) In multiplication and division: The number of significant figures which can be relied on in a product or quotient is never greater than the number of reliable significant figures in the weakest factor.

The relative error of a product or quotient may be as great as the sum of the relative errors of the separate items.

(4) In powers and roots: The relative error in the nth power of a number is n times the relative error in the number itself. Similarly, the relative error in $\sqrt[n]{x}$ is only $1/n$th of the relative error in x itself.

(5) In exponents and logarithms: If $y = e^x$, or $x = \log y$, then an absolute error of say .01 in x corresponds approximately to a relative error of .01 in y.

(6) In arithmetic or geometric mean: The relative error in the arithmetic or geometric mean of a number of quantities will be approximately the same as the relative error of the individual items (greater than the least of these relative errors and less than the greatest of them).

Rejection of superfluous figures. It is a fundamental rule of computation that a result should never be stated to a greater degree of precision than is justified by the data. All superfluous digits are misleading and should be rejected from the result.

If the first rejected figure is 5 or more, the preceding figure should be increased by one; otherwise, it should be left unchanged.[1]

For example, 3.14159 reduced to four figures is 3.142. Again, 6.1297 reduced to four figures is 6.130. Note that in a decimal fraction a final zero is as significant as any other final digit in determining the degree of precision. But in the case of a whole number like 3140000 the final zeros leave the reader in doubt whether the number of reliable significant figures is 3, 4, 5, 6, or 7. This ambiguity can be removed by writing the number in the form 3140000, or 3140000, or 3140000, etc., as the case may require; or, more usually, in the form 3.14×10^6, or 3.140×10^6, or 3.1400×10^6, etc., as the case may require.

This latter "notation by powers of 10" should always be used in the case of very large or very small numbers. For example,

$$0.000003140 = 3.140 \times 10^{-6}.$$

(*Note:* In this notation, the exponent of the power of 10 is the same as the "characteristic" of the logarithm of the number.)

DEFINITIONS OF VARIOUS KINDS OF MEANS OR AVERAGES

(1) The **arithmetic mean** (AM) of n numbers, $x_1, x_2, \cdots x_n$, is $1/n$th of their sum:

$$AM = \frac{1}{n}(x_1 + x_2 + \cdots + x_n), \text{ or } AM = \frac{1}{n}\Sigma x_i.$$

The AM is what is ordinarily meant when the term "mean" or "average" is used without further qualification. It is related to the **center of gravity** (or centroid) in mechanics, the center of gravity of a set of n equal particles being a point whose distance from any fixed plane is the AM of the distances of the several particles from that plane. It is also related to the "**method of least squares,**" since the sum of the squares of the deviations of the n numbers from any value X is a minimum when X is the AM of the numbers.

In computing an AM note that adding any constant, $\pm h$, to all the numbers has the effect of adding $\pm h$ to their AM.

For two numbers, a and b, the $AM = \frac{1}{2}(a + b)$.

(2) The **geometric mean** (GM) of n (positive) numbers, $x_1, x_2, \cdots x_n$, is the nth root of their product:

$$GM = \sqrt[n]{x_1, x_2, \cdots x_n}.$$

[1] A refinement of this rule is sometimes to be recommended, namely: if the rejected figures are exactly 5000 $\cdots$, the preceding figure should be raised when it is odd and left unchanged when it is even.

In computing the GM of n numbers, it is usually convenient to use the formula:

$$\log(GM) = \frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n), \text{ or } \log(GM) = \frac{1}{n}\Sigma(\log x_i),$$

that is, take the AM of the logarithms of the numbers, and then take the anti-log of the result.

For two numbers, a and b, the GM is $x = \sqrt{ab}$. This is called also the mean proportional between a and b, since $a : x = x : b$. By drawing a semicircle on $a + b$ as diameter, the value of x can be constructed geometrically, as in Figure 1.

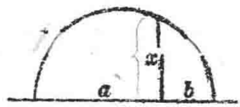

FIG. 1

(3) The **harmonic mean** (HM) of n (positive) numbers, $x_1, x_2, \cdots x_n$, is the reciprocal of the arithmetic mean of the reciprocals of the numbers:

$$HM = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}, \text{ or } \frac{1}{HM} = \frac{1}{n}\Sigma\left(\frac{1}{x_i}\right).$$

The chief use of the HM is in averaging frequencies, $1/x$ being called a frequency when x is a duration.

For example, in steamship statistics, the average number of trips per year may be more significant than the average number of days spent on each trip.

For two numbers, a and b, $HM = \dfrac{2\,ab}{a + b}$.

The AM, GM, and HM are the so-called classical means, known to the Greeks.

(4) The **contra-harmonic mean** (CHM) is almost as old, but is of very slight importance to-day:

$$CHM = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \text{ or } CHM = \Sigma(x_i^2)/\Sigma(x_i).$$

(5) The **root-mean-square** (RMS) of n numbers, $x_1, x_2, \cdots x_n$, is the square root of the arithmetic mean of their squares:

$$RMS = \sqrt{\frac{1}{n}(x_1^2 + x_2^2 + \cdots + x_n^2)}, \text{ or } RMS = \sqrt{\frac{1}{n}\Sigma(x_i^2)}.$$

The RMS is related to the **radius of gyration** in mechanics, the radius of gyration of a set of n equal particles, with respect to a given axis, being the RMS of the radial distances of the several particles from that axis. In statistics, the RMS of the deviations of a set of numbers from their arithmetic mean is called the **standard deviation** (SD) of those numbers.

The standard deviation of a set of numbers is also equal to the *RMS* of the differences between the numbers taken two by two; thus, if x is the *AM* of the numbers, then

$$SD = \sqrt{\frac{1}{n}\sum_i (x_i - x)^2} = \frac{1}{n}\sqrt{\sum_{i,j}(x_i - x_j)^2},$$

where $n(n-1)/2 =$ the number of the differences in question.

For any positive numbers, $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, the order of magnitude of these five means is as follows (unless the numbers are all equal):

$$x_1 < HM < GM < AM < RMS < CHM < x_n.$$

For the special case of two numbers, a and b, the following facts may be noted:

The *GM* of two numbers is the *GM* between their *HM* and their *AM*.

The *AM* of two numbers is the *AM* between their *HM* and their *CHM*.

The *RMS* of two numbers is the *GM* between their *AM* and their *CHM*.

The following general formulas, due to Mr. R. M. Foster, may also be noted:

$$M = [(x_1^k + x_2^k + \cdots + x_n^k)/n]^{1/k}$$

$$M' = \frac{x_1^{k+1} + x_2^{k+1} + \cdots + x_n^{k+1}}{x_1^k + x_2^k + \cdots + x_n^k}$$

If $k =$	$-\infty$	-1	0	1	2	∞
then $M =$	x_1	*HM*	*GM*	*AM*	*RMS*	x_n
and $M' =$	x_1	*HM*	*AM*	*CHM*		x_n

(The proof involves the evaluation of certain simple indeterminate forms.)

(6) The **median** of a set of quantities is, roughly speaking, the middle one of the set, when they are arranged in order of magnitude (*i.e.* "arrayed"). If the number of quantities is even, and the two middle quantities are not equal, the median is commonly taken as the number halfway between them. More exactly, the median, in this case, is a number X uniquely determined by the equation

$$(X - a_1)(X - a_2) \cdots (X - a_k) = (a_{k+1} - X)(a_{k+2} - X) \cdots (a_n - X),$$

where $a_1, a_2, \cdots a_k$ are the quantities of the lower half, and $a_{k+1}, a_{k+2}, \cdots a_n$, the quantities of the upper half of the set. (D. Jackson, *Bull. Amer. Math. Soc.*, Jan. 1921.)

The sum of the absolute deviations of n numbers from any value X is a minimum when X is the median of those numbers.

(7) The mode of a set of quantities is that quantity which occurs most often (*i.e.* is the most fashionable), if such a quantity exists. Any quantity which occurs more often than any other quantity near it in size may be called a relative mode (or simply *a* mode) of the set.

A general mathematical formula including the arithmetic mean, the median, and the mode is due to D. Jackson and R. M. Foster: Let X be the value of x which minimizes $\Sigma|x_i - x|^p$. Then if $p = 2$, $X =$ the arithmetic mean; if $p \to 1$, $\lim X =$ the median; if $p \to 0$, $\lim X =$ the mode. We note also that if $p \to \infty$, $\lim X = \frac{1}{2}(x_1 + x_n)$, where x_1 is the smallest and x_n the largest of the given quantities.

In the case of the median and the mode (as in the case of the AM), adding a constant, $\pm h$, to all the numbers has the effect of adding the same constant to the mean. (This is not true in case of the other four types of means.)

The following general properties are often useful:

In the case of any one of the seven means, multiplying all the numbers by a constant factor, c, has the effect of multiplying the mean by the same constant, c. ("Change of scale.")

In computing the AM, GM, HM, or RMS of n numbers, it is allowable, after grouping the numbers in any way, to replace each number of any group by the corresponding mean of that group. (This is not allowable in the case of the CHM, the median, or the mode.)

Weighted means. If the given numbers $x_1, x_2, \cdots x_n$ have different degrees of importance, as indicated by "weights" $w_1, w_2, \cdots w_n$, then we may speak of the **weighted mean** of these numbers (of any one of the seven kinds). Any kind of weighted mean of the given set of n numbers is defined as the corresponding kind of simple mean of a set of W numbers, in which x_1 occurs w_1 times, x_2 occurs w_2 times, etc., and $W = w_1 + w_2 + \cdots + w_n$ is the sum of the weights.

For example, the weighted arithmetic mean is $\frac{1}{W}(w_1x_1 + w_2x_2 + \cdots + w_nx_n)$; the weighted geometric mean is $(x_1^{w_1}x_2^{w_2} \cdots x_n^{w_n})^{1/W}$; etc.

The term "weighted mean," or "weighted average," used without qualifying adjective, usually indicates the weighted arithmetic mean.

PERMUTATIONS AND COMBINATIONS. THE BINOMIAL THEOREM

Permutations. The number of possible permutations or arrangements of n different elements is "n factorial" $= n! = 1 \cdot 2 \cdot 3 \cdots n$. Another notation is $\underline{|n} = n!$

Thus, the three letters a, b, c admit $3! = 6$ permutations: *abc, acb, bac, bca, cab, cba.*

If among the n elements there are p equal ones of one sort, q equal ones of another sort, r equal ones of a third sort, etc., where $p + q + r + \cdots = n$, then the number of possible permutations is

$$(n!) / [(p!)(q!)(r!) \cdots].$$

Thus, the four letters a, b, b, b, admit $24/[(1)(6)] = 4$ permutations: *abbb, babb, bbab, bbba.*

Combinations. The number of possible combinations or groups of n elements taken r at a time (without repetition of any element within any one group) is ${}_nC_r = \dfrac{n!}{(n-r)!\,r!}$ = the coefficient of the term in x^r in the binomial expansion of $(1 + x)^n$. (Notice that ${}_nC_r = {}_nC_{n-r}$).

Thus, the five letters *abcde* taken two at a time give ${}_5C_2 = 10$ combinations: *ab, ac, ad, ae, bc, bd, be, cd, ce, de.*

If repetitions are allowed within each group, then the number of combinations of n elements taken r at a time is ${}_{n+r-1}C_r$.

Thus, five letters taken two at a time, repetitions allowed, give ${}_6C_2 = 15$ combinations: *aa, ab, ac, ad, ae, bb, bc, bd, be, cc, cd, ce, dd, de, ee.*

The general principle underlying the theory of permutations and combinations is this: If we can do one thing in m ways and another thing in n ways, then we can do both things together in mn ways.

The binomial theorem. If n is any positive integer,

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{1\cdot 2}p^{n-2}q^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}p^{n-3}q^3 + \cdots + q^n$$

$$= p^n + {}_nC_1p^{n-1}q + {}_nC_2p^{n-2}q^2 + {}_nC_3p^{n-3}q^3 + \cdots + q^n,$$

where ${}_nC_1 = n$, ${}_nC_2 = [n(n-1)]/(2!)$, ${}_nC_3 = [n(n-1)(n-2)]/(3!)$, $\cdots$

$${}_nC_r = [n(n-1)(n-2)\cdots(n-r+1)]/(r!)$$

are the **binomial coefficients.**

Note that ${}_nC_{n-r} = {}_nC_r = \dfrac{n!}{(n-r)!\,r!}$.

Other notations are ${}_nC_r = \binom{n}{r} = (n)_r$.

TABLE OF BINOMIAL COEFFICIENTS

n	${}_nC_0$	${}_nC_1$	${}_nC_2$	${}_nC_3$	${}_nC_4$	${}_nC_5$	${}_nC_6$	${}_nC_7$	${}_nC_8$	${}_nC_9$	${}_nC_{10}$
1	1	1	..	..	..	..	..	..	..	..	..
2	1	2	1	..	..	..	..	..	..	..	..
3	1	3	3	1	..	..	..	..	..	..	..
4	1	4	6	4	1	..	..	..	..	..	..
5	1	5	10	10	5	1	..	..	..	..	..
6	1	6	15	20	15	6	1	..	..	..	..
7	1	7	21	35	35	21	7	1	..	..	..
8	1	8	28	56	70	56	28	8	1	..	..
9	1	9	36	84	126	126	84	36	9	1	..
10	1	10	45	120	210	252	210	120	45	10	1
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.

Note that each number, plus the number on its left, gives the number next below.

STIRLING'S FORMULA. THE BERNOULLI NUMBERS

Stirling's formula for *n* factorial. The following formula gives a good approximation to $n!$ for large values of n:

$$n! = (\sqrt{2\pi n})(n^n)(e^{-n}), \text{ or, more accurately,}$$

$$n! = (\sqrt{2\pi n})(n^n)(e^{-n})(e^{\frac{\theta}{12n}}), \text{ where } 0 < \theta < 1,$$

whence

$$\log_e(n!) = (n + \tfrac{1}{2})\log_e n - n + \log_e(\sqrt{2\pi}) + \frac{\theta}{12n},$$

$$\text{or } \log_{10}(n!) = (n + \tfrac{1}{2})\log_{10} n - (.434294482\,n) + .39909 + \frac{.03619\,\theta}{n}.$$

The last term, in which $0 < \theta < 1$, indicates the degree of approximation attained. For example, if $n = 1000$, $\log_{10}(1000!) = 2567.6046$, so that $1000! = 4.024 \times 10^{2567}$.

A seven-place table of $\log_{10}(n!)$ up to $n = 1000$ is given in *Pearson's Tables*, page 98, and in *Glover's Tables*, page 482.

A still more accurate approximation is

$$\log_e(n!) = (n + \tfrac{1}{2})\log_e n - n + \log_e(\sqrt{2\pi}) + \frac{B_1}{1\cdot 2}\frac{1}{n} - \frac{B_3}{3\cdot 4}\frac{1}{n^3} + \frac{B_5}{5\cdot 6}\frac{1}{n^5} - \frac{B_7}{7\cdot 8}\frac{1}{n^7} + \frac{\theta B_9}{9\cdot 10}\frac{1}{n^9},$$

where $B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = \frac{1}{42}$, $\cdots$ are the Bernoulli numbers (see below), and $0 < \theta < 1$.

Wallis's formula for π. Wallis's formula (useful in connection with the proof of Stirling's formula) is an infinite product the limit of which is $\pi/2$:

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots\frac{2n}{2n-1}\,\frac{2n}{2n+1}\cdots$$

The Bernoulli numbers. The following numbers occur in the expansion of many functions, such as $\tan x$, $\sec x$, $x/(e^x - 1)$, etc.

$B_1 = 1/6$	$B_2 = 1$
$B_3 = 1/30$	$B_4 = 5$
$B_5 = 1/42$	$B_6 = 61$
$B_7 = 1/30$	$B_8 = 1385$
$B_9 = 5/66$	$B_{10} = 50521$
$B_{11} = 691/2730$	$B_{12} = 2702765$
$B_{13} = 7/6$	$B_{14} = 199360981$
$B_{15} = 3617/510$	$B_{16} = 19391512145$
$B_{17} = 43867/798$	$B_{18} = 2404879675441$
$B_{19} = 174611/330$	$B_{20} = 370371188237525$
etc.	etc.

The numbers $B_2, B_4, B_6, \cdots$ are sometimes denoted by $E_2, E_4, E_6, \cdots$ or by $E_1, E_2, E_3, \cdots$; while the numbers $B_1, B_3, B_5, \cdots$ are sometimes denoted by $B_2, B_4, B_6, \cdots$ or b $B_1, B_2, B_3, \cdots$

For recursion formulas, see B. O. Peirce, *Table of Integrals*. For an extended table, see *Glover's Tables*. For large values of n, the following approximations are useful:

$$\frac{B_{2n-1}}{(2n)!} = \frac{2}{(2^{2n}-1)\pi^{2n}}\left[1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \cdots\right]$$

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}}\left[1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \cdots\right]$$

THE GAMMA FUNCTION

The Gamma Function of any positive number n is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx.$$

If n is a positive integer, $\Gamma(n+1) = n!$. (See Stirling's formula, above.) In general, $\Gamma(n+1) = n\,\Gamma(n)$, so that the value of $\Gamma(n)$ for any positive n can be found, by successive reductions, from a table covering the range from any integer to the succeeding integer, as, for example, from $n = 1$ to $n = 2$. In particular,

$$\Gamma(0) = \infty,\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi},\ \Gamma(1) = 1,\ \Gamma(2) = 1,\ \Gamma(3) = 2.$$

The graph of the function is shown in Figure 2. The minimum point is given by $\Gamma(1.4616321) = .8856032$. Tables of the Gamma Function are given in *Pearson's Tables* and in *Glover's Tables.*

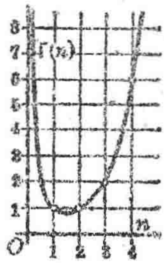

FIG. 2

The Beta Function. The Beta Function of any two positive numbers, m and n, is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

The hypergeometric series. The hypergeometric series is a function of x involving three parameters, a, b, c:

$$F(a, b, c, x) = 1 + \frac{a \cdot b}{1 \cdot c}x + \frac{a(a+1)}{1 \cdot 2}\,\frac{b(b+1)}{c(c+1)}x^2 + \frac{a(a+1)(a+2)}{1 \cdot 2 \cdot 3}\,\frac{b(b+1)(b+2)}{c(c+1)(c+2)}x^3 + \cdots$$

$$= \frac{\Gamma(c)}{\Gamma(a) \cdot \Gamma(c-a)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt.$$

GAUSS'S NORMAL ERROR CURVE, OR PROBABILITY CURVE

Constants of the normal curve. The most important constants connected with the normal curve of error are the following (see Figure 3):

y_0 = maximum ordinate (where $x = 0$), or height at the "mode."

A = total area, from $x = -\infty$ to $x = +\infty$. If the curve is given by a finite number of equi-spaced ordinates, then, approximately, $A = N \cdot \Delta x$, where N = total length of the ordinates ("total population"), and Δx = distance between the ordinates ("class interval").

σ = "standard deviation" or "root-mean-square error" = abscissa of point of inflection, given by

$$\sigma = \sqrt{\frac{1}{A}\int_{-\infty}^{+\infty} x^2 y dx}, \text{ or, approximately, } \sigma = \sqrt{\frac{1}{N}\Sigma(x^2 y)}.$$

p = "probable error" = value of the abscissa such that the area from $x = -p$ to $x = +p$ is half the total area A. Here $p = (\rho\sqrt{2})\sigma = 0.674489749\,\sigma$, where $\rho = 0.476936276 \cdots$ is a number defined by the equation

$$\frac{2}{\sqrt{\pi}}\int_0^\rho e^{-t^2}dt = \frac{1}{2}.$$