Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Introduction

This is the second in a chain of (hopefully) three volumes devoted to an explication of the fundamentals of semialgebraic topology over an arbitrary real closed field R. We refer the uninitiated reader to the preface of the first volume [LSA][1] and some other papers cited there to get an idea of the program we have in mind with the term "semialgebraic topology" as a basis of real algebraic geometry.

Let us roughly recall what has been achieved in the first volume and where we stand now.
As we explained in [LSA], the "good" locally semialgebraic spaces, which fortunately seem to suffice for most applications, are the regular paracompact ones. These are precisely those locally semialgebraic spaces which can be triangulated (I.4.8 and II.4.4)[2]. Moreover, any locally finite family of locally semialgebraic sets in such a space can be triangulated simultaneously (II.4.4). This fact seems to be the key result for many proofs in [LSA].

We accomplished less work on the triangulation of locally semialgebraic maps. Here our main result has been the triangulability of finite maps (II.6.13). Much more can probably be done, as is to be expected by the book [V] of Verona, but we do not pursue this line of investigation in the present volume. {Verona works over $\mathbb{R}$ and uses transcendental techniques.}

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1 cf. the references

2 This refers to Example 4.8 in Chapter I and Theorem 4.4 in Chapter II of [LSA]. The main body of this volume starts with Chapter IV. The signs I, II, III refer to the chapters of [LSA].

On the other hand we obtained in Chapter II of [LSA] a fairly detailed picture of the various possibilities how to "complete" a regular paracompact space M, i.e. to embed M densely into a partially complete regular paracompact space. Partial completeness is a typical notion of semialgebraic topology which has no counterpart in classical topology, cf. I, §6.

In Chapters I and II of [LSA] we also obtained basic results on the structure of locally semialgebraic maps. But the theory of fibrations and covering maps (= Überlagerungen) had to be delayed since a certain amount of homotopy theory is needed here, not yet available in the first two chapters.

Some of that homotopy theory has been presented in the last Chapter III of [LSA]. Our central result there are the two "main theorems" in various versions (III.3.1, 4.2, 5.1, 6.3, 6.4). As a consequence of these theorems all the homotopy groups and various homotopy sets in the category of regular paracompact spaces over R are "equal" to homotopy groups (resp. sets) in the classical topological sense of such spaces over $\mathbb{R}$. This opens the possibility to transfer a considerable amount of classical homotopy theory to the locally semialgebraic setting, as has been illustrated in Chapter III by several examples.

The homotopy theory in [LSA] seems to be sufficient for studying (ramified) coverings of regular paracompact spaces. To some extent it also gives access to the theory of fibrations and fiber bundles for such spaces (although here something remains to be desired, see below). Nevertheless this homotopy theory has serious deficiencies compared with classical (= topological) homotopy theory, and this brings us to the contents of the present volume.

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The main deficiencies are the following.

1) In the category LSA(R) of regular paracompact locally semialgebraic spaces over R we do not have infinite CW-complexes at our disposal.
2) In LSA(R) we do not have mapping spaces Map(X,Y) and prominent subspaces of them, as for example loop spaces ΩX , at our disposal.

One main goal in the present volume is to explain how the first deficiency can be overcome. We will construct "semialgebraic" CW-complexes over the field R. A CW-complex over R is a ringed space over R [LSA, p. 3] which is a suitable inductive limit of "polytopes" over R, together with a cell structure. Such inductive limits will generally be called "weak polytopes". (We briefly alluded to these spacec at the end of III, §6 and in [DK_6].) By a polytope over R we simply mean a complete affine semialgebraic space over R. This terminology is justified since these spaces are precisely all ringed spaces over R which are isomorphic to the underlying semialgebraic space of some closed finite simplicial complex over R, hence isomorphic to the union of finitely many closed simplices in some R^n.

We have to be careful which inductive systems of polytopes we admit in building weak polytopes. This is a delicate problem. If we are too restrictive then our weak polytopes will not be useful. On the other hand, if we are too permissive then we are in danger that our inductive limits become too wild spaces. (Recall that every real closed field different from $\mathbb{R}$ is totally disconnected in the topological sense.) Working in the category of ringed spaces over R gives us control which continuous functions we admit on a given space, and this gives us control on connectedness and other geometric properties implicitly.

Once we have defined weak polytopes in the right way and have established the basic properties of these spaces it will be an easy matter to define cell structures on some of them, which will be our CW-complexes. Then the door is open to transfer a really big amount of classical homotopy theory to the semialgebraic setting. In particular we can define spectra, in the sense of algebraic topology, and generalized homology and cohomology theories over R, and we can work with them nearly as easily as in classical homotopy theory (cf. Chapter VI).

Although the category of weak polytopes suffices to deal with infinite CW-complexes it is technically advisable to work in a slightly broader category, the category WSA(R) of "weakly semialgebraic spaces" over R. These spaces are inductive limits of affine semialgebraic spaces instead of just polytopes. For example, an open subspace (in the sense of locally ringed spaces) of a weak polytope is a weakly semialgebraic space, but usually is not a weak polytope. It would be cumbersome to exclude open subspaces of weak polytopes from our considerations. WSA(R) contains the category LSA(R) of regular paracompact locally semialgebraic spaces over R as a full subcategory.

The morphisms between weakly semialgebraic spaces will be called weakly semialgebraic maps. In Chapter IV we give the definition and basic properties of weakly semialgebraic spaces and maps. The key result for later use seems to be that in the category WSA(R) a space M can be glued to another space N along any closed subspace A of M by a "partially proper" weakly semialgebraic map $f : A \to N$ (Theorem IV.8.6). An analogous result had been proved in II, §10 within the category LSA(R) for proper maps. But the class of partially proper maps is much bigger than the class of proper maps and more useful (cf. I, §5-§6 and IV, §5 below). Most important, if the space M above is a weak polytope then also A is a weak polytope and every weakly semialgebraic map $f : A \to N$ is partially proper.

In general a weakly semialgebraic space M cannot be triangulated. But M still is isomorphic to a "patch complex". This is a very weak substitute of a simplicial complex which nevertheless is sufficient for some homotopy considerations.

Roughly one obtains a patch complex if one work with arbitrary affine semialgebraic spaces instead of simplices. The theory of patch complexes and their use in homotopy theory is displayed in Chapter V. Also some applications to open coverings (= Überdeckungen) of weakly semialgebraic spaces are given in V, §3.

Chapter V reveals that weakly semialgebraic spaces are beautiful from a homotopy viewpoint. For example, the two main theorems on homotopy sets from Chapter III in [LSA] extend to these spaces (V, §5) and there holds a strong "Whitehead theorem", stating that every weak homotopy equivalence is a genuine homotopy equivalence (Th. V.6.10). It is this chapter where the reader, having mastered the foundational labours of Chapter IV, will find out that weakly semialgebraic spaces are easy to handle and in some sense better natured, since "tamer", than topological spaces.

On the other hand, from a more geometric viewpoint, weakly semialgebraic spaces can be ugly. We shall demonstrate this in IV, §4 and Appendix C by rather simple examples. Various nice geometric properties we are accustomed to from locally semialgebraic spaces, as for instance the curve selection lemma, fail for these spaces. We do not know whether a weakly semialgebraic space M can be completed, i.e. densely embedded into a weak polytope. We do not know either whether M contains a weak polytope which is a strong deformation retract of M. In contrast to locally semialgebraic spaces there does not always exist a space N over the field R_o of real algebraic numbers such that

M is isomorphic to the base extension (cf. IV, §2) N(R) of N (cf. end of IV, §4). But still we can prove (V, §7) that M is homotopy equivalent to such a space N(R), even with N a CW-complex over R_o. Much later, in Chapter VII, §7, we shall see that M is homotopy equivalent to a closed simplicial complex.

Under the mild restriction that the base field R is sequential, i.e. R contains a sequence of positive elements converging to zero, things are even better. Then there exists, for every weakly semialgebraic space M over R, a canonical homotopy equivalence $p_M : P(M) \to M$ with P(M) a weak polytope. The space P(M) will be defined in Chapter IV, §9. It has the same underlying set as M but a "finer" space structure than M. On the set theoretic level, p_M is just the identity of M.

The space P(M) is the inductive limit of the system of all polytopes contained in M. It seems to be a very natural "simplification" of the space M (simplification for some purposes). If M is locally semialgebraic and locally complete then P(M) coincides with the space M_{loc} defined in I, §7. But already if M is a semialgebraic subset of some R^n which is not locally closed in R^n then P(M) is not locally semialgebraic.

More generally, given a weakly semialgebraic map $f : M \to N$, we shall define in IV, §10 a weakly semialgebraic space $P_f(M)$ together with a weakly semialgebraic map $p_f : P_f(M) \to M$ (if R is sequential) which has the following universal property. The map $f \circ p_f$ is partially proper, and every weakly semialgebraic map $q : L \to M$ with $f \circ q$ partially proper factors uniquely through p_f. If N is the one-point space then $P_f(M) = P(M)$.

These spaces $P_f(M)$, and in particular the spaces P(M), will do good service in homotopy theory at various places. They are typical for the somewhat different flavour of semialgebraic homotopy theory compared with classical homotopy theory.

A particularly good instance to see how the spaces $P_f(M)$ and similar ones can be used and how the various techniques we have developed in Chapters IV and V fit together is the proof of Theorem V.6.8 on d-equivalences (instead of just weak homotopy equivalences) which precedes and implies the Whitehead theorem mentioned above. The reader cannot do better than trying to obtain an impression of the main lines of this proof at an early stage in order to get a good feeling for the subject.

Of course, we try to proceed in semialgebraic homotopy theory as much as possible in a way parallel to the classical topological homotopy theory, as long as this is advisable. Here there comes up a dichotomy of goals and methods everyone working in this area will face.

On the other hand, one would like to obtain results in the semialgebraic theory by transfer from the topological theory, as already exercised in Chapter III. One wants to have available the enormous body of results of topological homotopy theory in the semialgebraic setting without much further labour.

On the other hand, there is a more radical viewpoint, to the best of my knowledge first expressed by Brumfiel in his book [B]: One should do algebraic topology from scratch over an arbitrary real closed field in such a way that the field $\mathbb{R}$ does not play any special role.

This is an ambitious program. While writing this volume I somewhat oscillated between the two viewpoints. Whenever the semialgebraic geometry was easy I avoided transfer principles. When not I gave preference to the first view point, but often I also tried to indicate how things can be done in the spirit of the second one.

Long passages in Chapter V may nourish the conviction that a homotopy theory in the sense of Brumfiel is already at hands. But there are still problems to be settled. As a testing ground I have chosen here - as already in [LSA], Chapter III - the homotopy excision theorem of Blakers and Massey. In topology there exists an elementary proof of this theorem going back to Boardman, cf. [DKP, p. 211ff]. This proof (as well as the proof of Blakers and Massey) strongly uses the axiom of Archimedes in the field of real numbers. We are able to prove the analogue of the theorem for weakly semialgebraic spaces (V, §7), but for that we need the Blakers-Massey theorem for topological CW-complexes and transfer techniques.

The homotopy theory developed in Chapter V suffices for studying generalized homology and cohomology groups of pairs of weakly semialgebraic spaces. {The word "generalized" means that we do not insist on the Eilenberg-Steenrod dimension axiom.} In Chapter VI we define generalized homology and cohomology theories on the category $\mathcal{P}(2,R)$ of weak polytopes over R in full analogy to the definition of such theories on the category $\mathfrak{W}(2)$ of pairs of topological CW-complexes $[W_2]$ (or [W], [Sw], etc.). We then explicate how every topological homology theory h_* or cohomology theory h^* on $\mathfrak{W}(2)$ leads in a natural way to a homology theory respectively cohomology theory on $\mathcal{P}(2,R)$ which we denote again by h_* resp. h^*. We thus obtain a bijection, up to natural equivalence, between the homology and cohomology theories on $\mathfrak{W}(2)$ and on $\mathcal{P}(2,R)$ for R fixed (VI, §2-4). We extend these theories in VI, §5 from $\mathcal{P}(2,R)$ to the category WSA(2,R) of pairs of weakly semialgebraic spaces over R, and we prove in VI, §6 a fairly general excision theorem for the groups $h_n(M,A)$

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and $h^n(M,A)$. We also describe the theories h_* and h^* by spectra as one does in topology (VI, §8).

In this whole business it is important that we have weakly semialgebraic spaces at our disposal instead of just locally semialgebraic spaces. We mentioned already the need for infinite CW-complexes. But even suspensions pose a problem. They play an essential role in generalized homology theory, of course. Unfortunately we do not have suspensions for arbitrary weakly semialgebraic spaces but only for weak polytopes. This turns out to be sufficient. But if M is a locally semialgebraic (pointed) weak polytope then usually the suspension SM will not be locally semialgebraic.

If h_* is one of the prominent homology or cohomology theories in topology, as singular homology, singular cohomology, orthogonal, unitary, or symplectic K-theory, one of various cobordism theories, then there remains the important task to attach a geometric meaning to the elements of $h_n(M,N)$ or $h^n(M,A)$ for (M,A) a pair of weakly semialgebraic spaces. {In topology usually such a meaning is inherent in the definition of these groups.}

In the next volume [SFC] we shall solve this problem for the K-theories mentioned above. In the present one we solve it for ordinary homology $H_*(-,G)$ and ordinary cohomology $H^*(-,G)$ with coefficients in some abelian group G. These are those homology and cohomology theories which fulfill the Eilenberg-Steenrod dimension axiom. They arise from topological singular homology and cohomology theory with coefficients in G.

We prove in VI, §3 that if (M,A) is a pair of CW-complexes then the groups $H_n(M,A;G)$ and $H^n(M,A;G)$ have a description by cellular chains and cochains as in topology. It is then easy to conclude that for (M,A) a pair of locally semialgebraic spaces, these groups coincide with the groups $H_n(M,A;G)$ and $H^n(M,A;G)$ defined essentially by Delfs [D], [D_1], [DK_3]. {We described the groups $H_n(M,A;G)$ in III, §7.}

Here our theory reaches a remarkable point. To understand, why, let us recall the approach of Delfs to the homology groups, say, of a single polytope M. {We take $A = \emptyset$.} The polytope M can be triangulated. Choosing an isomorphism $\varphi : |K|_R \xrightarrow{\sim} M$ with K a finite abstract simplicial complex we "know" a priori what $H_n(M,G)$ should be: It should coincide with the abstract homology $H_n(K,G)$ of the simplicial complex K. The problem is, to prove that the groups $H_n(K,G)$ do not depend on the choice of the triangulation. Delfs solves this problem in an ingenious way. He looks at the simplicial cohomology groups $H^n(K,G)$ for the triangulations of M. He proves that they all are naturally isomorphic to the cohomology groups $H^n(M,G_M)$ of the constant sheaf G_M with stalk G. Knowing that the $H^n(K,G)$ are independent of the triangulation he concludes that the $H_n(K,G)$ also are independent of the triangulation.

In the course of this approach Delfs has to cope with some tedious geometric problems. {The main task is to prove the homotopy invariance of the groups $H^n(M,G_M)$. In [D_1] Delfs solves this problem brilliantly by using sheaf theory on abstract locally semialgebraic spaces.} The remarkable fact now is that we obtain the independence of the groups $H_n(K,G)$ from the choice of the triangulation in a much easier way. Once we have the homotopy theory of Chapter V at hands, which is a straightforward matter, we define the ordinary homology groups $H_n(M,G)$ almost by general categorial nonsense, and prove $H_n(M,G) \cong H_n(K,G)$ in the standard way (cf. VI, §3). Thus one may say that it is possible to circumvent the labours of Delfs by enlarging the category of affine semialgebraic spaces over R to a category of spaces which is more comfortable for homotopy considerations, namely WSA(R). {But notice that our approach does not give a connection of ordinary cohomology with sheaf cohomology.}

How about an interpretation of the elements of $H_n(M,A;G)$ by chains of singular simplices, as in topology? Of course, a singular simplex here means a semialgebraic map (= morphism) from the closed standard simplex $\nabla(n)$ in R^{n+1} to M. For any pair (M,A) of weakly semialgebraic spaces over R we can define the singular chain complex $C.(M,A;G)$ as in topology. The problem is to prove that the groups $H_n(C.(M,A;G))$ fit together to an ordinary homology theory and that $H_o(C.(*,\emptyset;G)) \cong G$, with $*$ denoting the one point space. This would imply a natural isomorphism from this homology theory to $H_*(-,G)$.

Delfs and I have tried for years in vain to find such a proof in a direct geometric way. The difficulty was always to prove an excision theorem for the groups $H_n(C.(M,A;G))$ in the case that the field R is not archimedean. We could not prove excision even for a triad of polytopes. As in classical theory one would like to make a given singular chain "small" with respect to a given finite open covering (with two open semialgebraic sets) by applying some iterated subdivision to the singular simplices in the chain. But the trouble is that, as long as one tries barycentric subdivision or some other sort of finite linear subdivision, the simplices have no reason to become small if R is not archimedean.

The last Chapter VII of the present book contains a solution of the problem - along very different lines. This solution is perhaps the most convincing single issue, up to now, to demonstrate that weakly semialgebraic spaces are really useful.

We proceed roughly as follows. Every simplicial set K (= semisimplicial set = semisimplicial complex, in other terminologies) can be "realized" as a weak polytope $|K|_R$ over R in much the same way as this is known in topology [Mi_1]. The space $|K|_R$ carries a natural structure of a CW-complex. If (K,L) is a pair of simplicial sets (of course, with L a simplicial subset of K), then it follows from the cellular description of ordinary homology mentioned above that the ordinary homology groups $H_n(|K|_R, |L|_R; G)$ can be identified with the well known (cf. [La] or [May]) "abstract" homology groups $H_n(K,L;G)$.

If M is a weakly semialgebraic space over R we can form the singular simplicial set Sin M consisting of the singular simplices of M. The realization $|\text{Sin}\, M|_R$ comes with a canonical weakly semialgebraic map $j_M : |\text{Sin}\, M| \to M$. We prove that j_M is a homotopy equivalence (VI, §7) following the book [LW] of Lundell and Weingram. {In topology j_M is only a weak homotopy equivalence. In most texts on simplicial methods - but non in [LW] - this is proved by already using the fact that the topological singular homology groups form an ordinary homology theory.}

More generally, if A is a subspace of M, then j_M gives a homotopy equivalence from the pair $(|\text{Sin}\, M|_R, |\text{Sin}\, A|_R)$ to (M,A). Thus

$$H_q(M,A;G) \cong H_q(|\text{Sin}\, M|_R, |\text{Sin}\, A|_R; G) \cong H_q(\text{Sin}\, M, \text{Sin}\, A; G),$$

and this group is $H_q(C.(M,A;G))$ by definition.

Since we know that the canonical maps j_M are homotopy equivalences the door is now wide open for the use of simplicial sets in semialgebraic geometry. Thus, finally, we can abolish our previous verdict "no simplicial sets, only simplicial complexes" [DK_3, p. 124].

Simplicial sets have proved to be enormously useful in many branches of topology, in particular in the theory of fibrations. Much of this material can now be used in semialgebraic geometry. Some applications to the theory of semialgebraic fibrations will be given in the next volume [SFC].

But one needs more. One needs simplicial spaces instead of just simplicial sets. By a simplicial space X over R we mean a simplicial object in WSA(R), i.e. a sequence $(X_n | n \in N_o)$ of weakly semialgebraic spaces over R with various weakly semialgebraic face and degeneracy maps between them (VII, §1). Simplicial sets may be regarded as discrete simplicial spaces over R.

Roughly half of our last Chapter VII is devoted to an explication of the fundamentals of simplicial spaces and their realizations. Difficulties for future application will arise from the fact that we are only able to construct the realization $|X|_R$ of a partially proper simplicial space X. By this we mean a simplicial space all whose face maps are partially proper. Fortunately discrete simplicial spaces are partially proper.

A reader having worked through the fundamentals of weakly semialgebraic spaces and maps in Chapter IV may feel bored to meet in Chapter VII similar stuff about simplicial spaces. To give such a reader some comfort we indicate now by an example that this stuff is really useful.

Let G be a complete semialgebraic group over R. {For instance think of some orthogonal group O(n,R).} If M is an affine semialgebraic space, then it is clear from the beginnings of semialgebraic geometry what is meant by a principal G-fibre bundle $\varphi : E \to M$ over M. The definition is exactly as in topology, of course with a finite trivializing covering of M by open semialgebraic subsets.

We now pose the following problem. Let S be a real closed overfield of

R and let $\psi : F \to M(S)$ be a principal $G(S)$-bundle over $M(S)$. Does there exist a principal G-bundle $\varphi : E \to M$ over M such that the base extension $\varphi_S : E(S) \to M(S)$ is isomorphic to ψ over $M(S)$?

It seems hard to solve this problem in a direct geometric way. We shall solve it in [SFC] in the affirmative as follows. Let $\mathcal{N}G$ denote the nerve of the group G. This is a simplicial space built as in topology, cf. Example VII.1.2.v below. $\mathcal{N}G$ is partially proper since G is complete (partially complete would suffice). Let BG denote the realization $|\mathcal{N}G|$. One finds as in topology that the isomorphism classes of G-principal bundles over M are in natural one-to-one correspondence with the elements of the homotopy set [M,BG]. By the first main theorem on homotopy sets the base extension map from [M,BG] to [M(S),(BG)(S)] is bijective (V.5.2.i; essentially this is already clear from III.3.1). By the canonical nature of the definition of $\mathcal{N}G$ it is evident that (BG)(S) = B(G(S)). Thus we have a natural bijection from [M,BG] to [M(S)B(G(S))]. We conclude that the isomorphism classes of principal G-bundles over M correspond uniquely with the isomorphism classes of principal G(S)-bundles over M(S) by base extension. The answer to the question above is "Yes".

At first glance the present book might convey the impression that in semialgebraic geometry one now has a homotopy theory at hands which is as good and easy as the topological one. But this impression is deceptive. In order to destroy it I come back to the two deficiencies of the homotopy theory in [LSA] listed above. While the first one disappears in the category WSA(R), the second one (existence of mapping spaces) remains serious.

One would like to have good substitutes (or "models") of the presumably not existing mapping spaces and their prominent subspaces. In VI, §7 we define "pseudo-mapping spaces" and "pseudo-loop spaces" which do <u>some</u>

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